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Vibration Isolation for a Drop Forge

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Table of Contents

List of Symbols

A	Generic Coefficient
a	Acceleration
B	Generic Coefficient
c	Damping Coefficient
c_c	Critical Damping Coefficient
e	Exponential
F	Force
$F_{t\,max}$	Max Force Transmission
λ	Gamma (Roots Coefficient)
k	Spring Constant
m	Mass
ϕ	Phase Angle
R	Ratio of Force Transmission
r	Roots Coefficient
t	Time
V_0	Initial Velocity
ω_n	Natural Frequency
ζ	Damping Ratio

List of Figures

List of Tables

Abstract

A drop forge which consists of a frame, hammer, damping pad and an anvil is to be analyzed to study the behavior of the vibration in the system when the damping pad is structured using four different damping situations; undamped, underdamped, overdamped, and critically damped. This allows the efficiency at differing damping coefficients to be examined. Deriving the governing dynamic equation of the system where the natural frequency and the damping ratio are explicitly presented and then using linear homogenous ordinary differential equation methods to obtain expressions for the force transmitted from the hammer, through the anvil, into the damping pad and into the floor for each case give the baseline to which the efficiency is found. By obtaining a ratio of the force transmitted in the undamped case to all other cases and comparing to a variety of damping ratios, we find the most efficient damping ratio that minimizes the force transmitted to be 0.26. This yields an expression of the damping coefficient, c, to be $0.52m\omega_n$. The percent decrease from an undamped system to the optimally damped system is found to be approximately 18.98 percent.

Introduction/Assumptions

We consider a drop forge which consists of a frame, hammer, damping pad and an anvil. The drop forge is used to shape work pieces through the transmission of force. The work piece to be formed is placed on the anvil and is struck by the hammer:

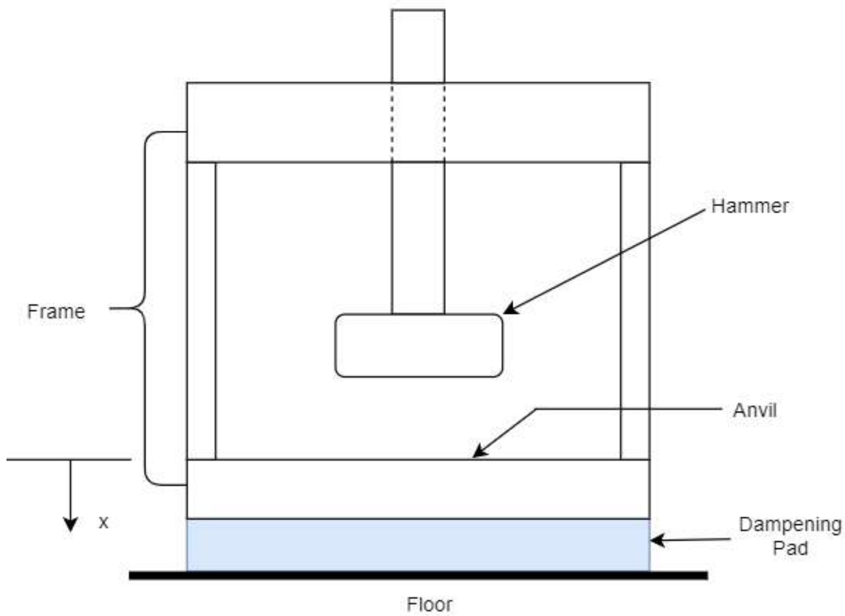

Figure 1: Drop Forge Schematic

When the hammer is raised, it is supported by a separate structure that is not shown. As the hammer falls, the impact force imparts an initial velocity, V_o, to the structure. The forge is mounted on a damping pad which could be made from various materials such as commercial shock mounts that consist of spring and damper elements that are arranged in parallel. As the force is transmitted, a vibration is transmitted to the floor. The effectiveness of damping to reduce the vibrational transmission to the floor is to be determined.

The efficiency of the damping can be analyzed through calculation of the damping coefficient, c, as a function of the mass of the forge and the spring constant of the pad, which minimizes the peak force transmitted to the floor. The percentage decrease in the peak force transmitted to the floor with optimum damping over that for a pad with the same spring constant and no damping will also be determined. Results will be displayed using a curve of the transmitted force as a function of the damping ratio, ζ.

To accurately solve this problem, we must consider a few assumptions. First, we assume that the mass of the hammer is negligible compared to the mass of the anvil. As shown in Figure 1, the positive x axis reference point is at the face of the anvil and continues down in the vertical direction towards the floor, so any positive forces or positions will be in the downward direction where negative forces or positions will be in the upward direction. The horizontal axis is not considered in the problem as we are assuming a linear system.

In this mechanical system, we consider Newtons Second Law, $F = ma$, which translates to the total force that is applied to the body is equal to the motion of the body. In the system, we assume that the reference point is also at the equilibrium point which eliminates the force created by the gravity because the equilibrium point is the point where the restorative force of the system opposes the force of the weight of the system (mass x gravity).

Other assumptions made are that we will not consider temperature, or any external forces, the anvil will not see any deformation from the force transmitted from the hammer, and the spring is ideal, meaning it will behave in a linear fashion. The system being considered will take the form $x'' + ax' + bx = 0$ since we have a linear homogeneous system where a and b are considered to be constant.

To analyze the problem at hand, we will be studying a few different systems; undamped, overdamped, critically damped, and underdamped. With an undamped system, we assume that the natural damping of the system is so small, and the motion of the system is considered over a short period of time that we can disregard the dampening all together. When considering the damped systems, we assume that the damping force is proportional to the velocity since the velocities will be small.

Conclusions

The purpose of this report is to analyze the efficiency of a drop forge system with a damping pad of differing damping coefficients. The hammer of the forge transmits a force through the anvil, into the damping pad and to the floor below. Depending on the damping coefficient and ratio chosen for the system, the vibrations seen can vary significantly.

The dynamic governing equation for the system, as shown in the analysis is found to be a linear homogeneous ODE

$$\boldsymbol{x'' + 2\zeta\omega_n x' + {\omega_n}^2 x = 0}$$

Using this equation and the differing situations of the damping coefficients

1. Undamped : $\zeta = 0$
2. Over-Damped : $\zeta > 1$
3. Critically Damped : $\zeta = 1$
4. Under-Damped : $\zeta < 1$

We derive expressions for the position, velocity, acceleration and as a result the max force transmission as outlined in the Tables 1, 2, and 3 below. Comparing the max force transmission of each case with the undamped case, we find our force transmission ratios. When compared to various damping ratios, we obtain a plot as shown in Figure 4.

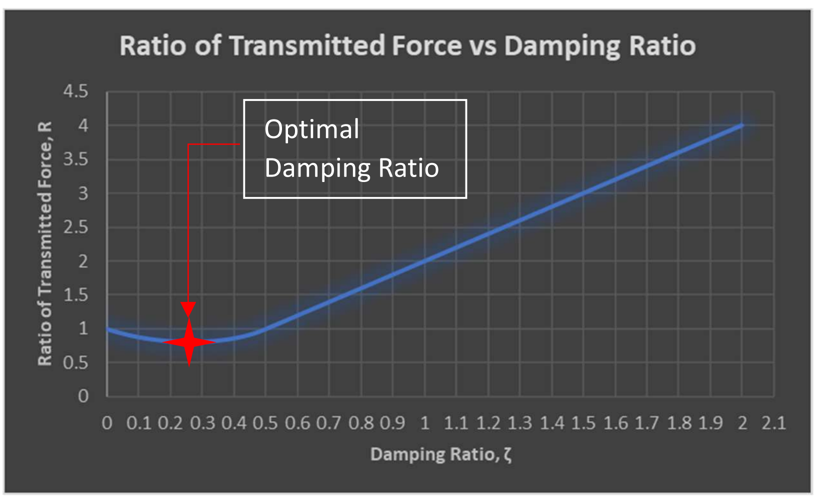

Figure 4: Plot of Ratio of Transmitted Force vs Damping Ratio

The optimum damping ratio is found at the minimum peak of the curve. This shows the damping ratio that minimizes the force transmission through the floor. This damping ratio is found to be equal to 0.26.

With the damping ratio of 0.26, knowing that the damping ratio is a ratio of the actual damping and the critical damping we find the optimal damping coefficient to be $\boldsymbol{c = 0.52m\omega_n}$.

Finally the percent decrease in peak force transmitted to the floor with the optimum damping over that for a pad with the same spring constant and no damping is found to be approximately 18.98% with the ratio of force transmission of the undamped system to be 1 and the ratio of force transmission of the optimal system to be 0.810188.

Analysis

To analyze the system properly, we must model the diagram as shown below.

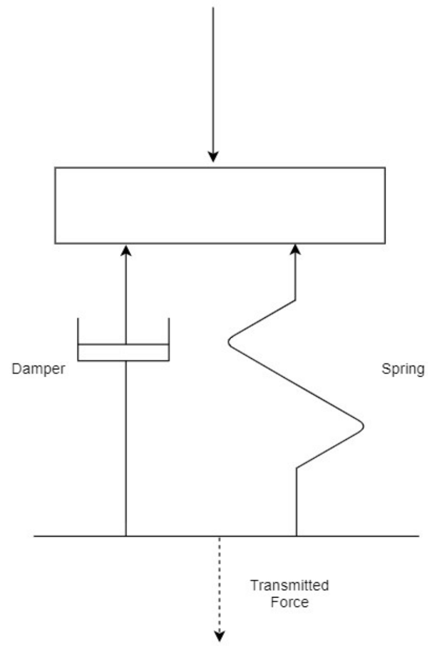

Figure 2: Simplified Diagram of System

This diagram of the system is general and certain components such as the damper will be neglected in certain cases such as the undamped case where damping will be zero but helps form a basis around which this problem can be depicted.

Foundational principles must be considered and understood when approaching a dynamic modeling problem. When considering a mechanical system with a represented spring, we must consider Hooke's Law which shows the relationship between the stiffness of a spring, denoted as the spring constant k, and the position, x, of the deformed spring and how they relate to the force of the spring.

$$\boldsymbol{F = -kx} \tag{1.0}$$

As mentioned in the introduction, Newtons Second Law will play a large roll in defining the mechanical dynamic governing equation which states that the total force applied to a body will be equal to the motion of the body. The motion of the body is defined by the mass of the body times the acceleration applied to the body.

$$\boldsymbol{F = ma} \tag{1.1}$$

We know that the acceleration is the second derivative of the position with respect to time, so we can rewrite Equation (1.1) as

$$\boldsymbol{F = mx'',\ x'' = \frac{d^2x}{dt^2} = Acceleration\ and\ m = Mass} \tag{1.2}$$

The total force on the system is the resultant of all the acting forces. The forces on the system are the restoration force of the spring, $-kx$ (per Hooke's Law), the force created by the mass and gravity, mg, and the force that opposes the gravitational pull, $-ks$, and the resistive force of the damping, $-cx'$, where c is known as the damping coefficient and is negative because it is acting against the motion of the system. As discussed in the assumptions, gravity in this case is not considered as the force opposing the gravitational pull negates the gravitational force, so we are left with $-kx$ and $-cx'$. With these forces considered, Equation (1.2) becomes

$$\boldsymbol{-kx - cx' = mx''} \tag{1.3}$$

Rearranging (1.3) yields the general governing equation of the system

$$\boldsymbol{mx'' + cx' + kx = 0} \tag{1.4}$$

where x'' is the acceleration, x' is the velocity, and x is the position.

To evaluate this ordinary differential equation, we must bring it into the standard form of a homogenous linear ODE, $x'' + p(t)x' + q(t)x = 0$. To do this, we divide all terms by the mass

$$\boldsymbol{x'' + \frac{c}{m}x' + \frac{k}{m}x = 0} \tag{1.5}$$

For our problem, our aim is to represent the governing equation with the natural frequency, ω_n and damping ratio, ζ. To accomplish this, we must define and manipulate the definitions of these two variables.

The damping ratio is utilized to express the damping level in a system relative to a critically damped system. The damping ratio is the ratio between the actual damping coefficient of the system, c, and the critical damping coefficient, $c_c = 2\sqrt{km}$

$$\boldsymbol{\zeta = \frac{c}{c_c}} \tag{1.6}$$

Frequency is the value of repeating occurrences in a set time span and is defined in cycles / second, $f = \frac{\omega_n}{2\pi}$. The natural frequency is the frequency that a system will oscillate without the presence of any external forces. In the mass spring system, the natural frequency is defined as

$$\boldsymbol{\omega_n = \sqrt{\frac{k}{m}}} \tag{1.7}$$

where manipulation yields

$$\boldsymbol{\frac{k}{m} = \omega_n{}^2} \tag{1.8}$$

We can represent the critical damping coefficient in terms of the natural frequency by dividing by the mass and relating equations (1.6) and (1.7)

$$\boldsymbol{c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n} \tag{1.9}$$

And $\zeta = \frac{c}{c_c}$ becomes $\frac{c}{2m\omega_n}$ and multiplying the denominator by ζ and dividing by m, yields

$$\boldsymbol{\frac{c}{m} = 2\zeta\omega_n} \tag{2.0}$$

Plugging Equations (1.8) and (2.0) into our standard form of the governing equations gives the derived general expression for the dynamic governing equation where the natural frequency and damping ratio are explicitly presented for all second order vibration problems

$$\boldsymbol{x'' + 2\zeta\omega_n x' + \omega_n{}^2 x = 0} \tag{2.1}$$

To fully study this second order vibration problem, we will analyze the four different vibration regimes that all depend on different damping ratios to find the optimal damping:

5. Undamped : $\zeta = 0$
6. Over-Damped : $\zeta > 1$
7. Critically Damped : $\zeta = 1$
8. Under-Damped : $\zeta < 1$

<u>Undamped System</u>

Starting with the Undamped system with a damping ratio of zero ($\zeta = 0$), we insert the damping ratio into our governing equation (2.1) and it is simplified to

$$\boldsymbol{x'' + \omega_n{}^2 x = 0} \tag{2.2}$$

We know through separation of variables a homogenous linear ODE of the form $x' + kx = 0$ has the general solution $x = Ce^{-rt}$ which is a good trial solution to use. To obtain the characteristic equation, we use the trial solution $x = Ce^{rt}$ and the second derivative x and plug it into our general equation

$$x'' = r^2 e^{rt}$$

$$r^2 e^{rt} + \omega_n{}^2 e^{rt} = 0$$

$$(r^2 + \omega_n{}^2)e^{rt} = 0$$

$$\boldsymbol{r^2 + \omega_n{}^2 = 0} \tag{2.3}$$

Now we find the roots of the characteristic equation to be $r = \pm i\omega_n$. These roots can be seen to be complex and with constant coefficients the general solution takes the form

$$\boldsymbol{x = Acos\omega_n t + Bsin\omega_n t} \tag{2.4}$$

To solve for the constants, we apply the first initial condition of zero position at time zero, $x(0) = 0$ and find that $A = 0$.

$$\boldsymbol{0 = Acos\big(\omega_n(0)\big) + Bsin(\omega_n(0))} \tag{2.5}$$

$$\boldsymbol{0 = A}$$

To solve for B, we plug A into equation (2.4) and take the derivative of the newly obtained equation which is equivalent to the velocity ($x' = V$). Here we apply another initial condition of zero initial velocity at time zero ($x'(0) = V_0$) and find $B = \frac{V_0}{\omega_n}$.

$$\boldsymbol{x = Bsin\omega_n t} \tag{2.6}$$

$$x' = B\omega_n cos\omega_n t$$

$$x'(0) = V_0 = B\omega_n cos\omega_n(0)$$

$$\boldsymbol{B = \frac{V_0}{\omega_n}}$$

Plugging our coefficients back into equation (2.4) and taking the first and second derivatives, we obtain our expression for the position (2.7), velocity (2.8) and acceleration (2.9).

$$\boldsymbol{x = \frac{V_0}{\omega_n} sin\omega_n t} \tag{2.7}$$

$$\boldsymbol{x' = V_o cos\omega_n t} \tag{2.8}$$

$$\boldsymbol{x'' = -V_o\omega_n sin\omega_n t} \tag{2.9}$$

Now that we have an expression for the acceleration, we can find the expression for the force transmitted using Newtons Second Law

$$\boldsymbol{F_T = -mx'' = m * V_o\omega_n sin\omega_n t} \tag{3.1}$$

Now that we have the expression for the transmitted force, we must determine the conditions for peak force by taking the derivative of the force with respect to time which can also be found by taking the third derivative of x and multiplying by the mass and is expressed as $-mx'''$.

$$\boldsymbol{\frac{dF_T}{dt} = -mx''' = m * V_o\omega_n{}^2 cos\omega_n t} = 0 \tag{3.2}$$

Therefore we can solve for $\omega_n t = \frac{n\pi}{2}, (n = 1,2,3..)$ and we can plug back into equation (3.1) gives the maximum transmitted force in the undamped case

$$\boldsymbol{F_{T_{Max}} = m * V_o\omega_n \sin\left(\frac{n\pi}{2}\right),\ n = 1, 2, 3..} \tag{3.3}$$

Overdamped System

Moving on to the overdamped system where the damping ratio is greater than 1, ($\zeta > 1$), we start with our dynamic governing equation, Eq 2.1 and obtain the characteristic equation

We know through separation of variables a homogenous linear ODE of the form $x' + kx = 0$ has the general solution $x = Ce^{-rt}$ which is a good trial solution to use. To obtain the characteristic equation, we use the trial solution $x = Ce^{rt}$ and the first and second derivative x and plug it into our general equation

$$x'' = r^2 e^{rt}$$

$$x' = re^{rt}$$

$$(r^2 + 2\zeta\omega_n r + \omega_n{}^2)e^{rt} = 0$$

$$\boldsymbol{r^2 + 2\zeta\omega_n r + \omega_n{}^2 = 0} \tag{3.4}$$

Now we find the roots of the characteristic equation which take the form $r_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b}, r_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b}$. The roots therefore simplify to

$$r = \frac{1}{2}(-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4(\omega_n{}^2)}$$

$$\boldsymbol{r_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}} \tag{3.5}$$

In the case of two real roots, a basis of solutions on any time interval is $x_1 = e^{r_1 t}, x_2 = e^{r_2 t}$. The general solution of the displacement yields

$$\boldsymbol{x = Ae^{(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})t} + Be^{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})t}} \tag{3.6}$$

To solve for the constants, we must apply our initial conditions again to Eq. 3.6 and its' derivative

$$\boldsymbol{x' = A(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})e^{(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})t} + B(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})e^{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})t}} \tag{3.7}$$

Applying the first initial condition of zero position at time zero, $x(0) = 0$.

$$x(0) = A + B = 0$$

And the second initial condition which is equivalent to the velocity ($x' = V$), we have the initial velocity at time zero ($x'(0) = V_0$)

$$x'(0) = V_0 = A(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}) + B\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right)$$

To solve for the coefficients, we can use a system of equations set up as

$$\begin{bmatrix} \boldsymbol{1} & \boldsymbol{1} \\ \boldsymbol{-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})} & \boldsymbol{-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{B} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{V_0} \end{bmatrix} \tag{3.8}$$

Where the inverse of the 2x2 matrix is

$$\begin{bmatrix} \frac{-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})}{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}) + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})} & \frac{-1}{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}) + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})} \\ \frac{\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})}{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}) + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})} & \frac{1}{(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}) + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1})} \end{bmatrix}$$

And multiplying both sides of the equation by the inverse yields the coefficients

$$\boldsymbol{A} = -\frac{\boldsymbol{V_0}}{\boldsymbol{-2\omega_n\sqrt{\zeta^2 - 1}}}, \boldsymbol{B} = \frac{\boldsymbol{V_0}}{\boldsymbol{-2\omega_n\sqrt{\zeta^2 - 1}}}$$

Next we need to plug the coefficients into our displacement, velocity, and acceleration equations which become

$$\boldsymbol{x = \frac{V_0}{2\omega_n\sqrt{\zeta^2-1}}\left(e^{\left(-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)t} - e^{\left(-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)t}\right)} \quad (3.9)$$

$$\boldsymbol{x' = \frac{V_0}{2\omega_n\sqrt{\zeta^2-1}}\left[\left(-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)e^{\left(-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)t} - \left(-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)e^{\left(-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)t}\right]} \quad (4.0)$$

$$\boldsymbol{x'' = \frac{V_0\omega_n}{2\sqrt{\zeta^2-1}}\left[\left(-\zeta+\sqrt{\zeta^2-1}\right)^2 e^{\left(-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)t} - \left(-\zeta-\sqrt{\zeta^2-1}\right)^2 e^{\left(-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)t}\right]} \quad (4.1)$$

We notice that in the exponent that we are negative, $-\zeta\omega_n$, then the maximum force will occur at time equal to zero because the curve of e to a negative power approaches zero as time goes positive and increases as time goes negative, but since we do not look at negative time, the time stamp of the maximum is at zero. Applying time of zero to the acceleration, we obtain

$$x''(0) = \frac{V_0\omega_n}{2\sqrt{\zeta^2-1}}\left[\left(-\zeta+\sqrt{\zeta^2-1}\right)^2 e^{\left(-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)t} - \left(-\zeta-\sqrt{\zeta^2-1}\right)^2 e^{\left(-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)t}\right]$$

$$x''(0) = \frac{V_0\omega_n}{2\sqrt{\zeta^2-1}}\left[-4\zeta\sqrt{\zeta^2-1}\right]$$

$$\boldsymbol{x''(0) = -2\zeta V_0\omega_n} \quad (4.2)$$

Plugging the acceleration into Newtons Second Law, Equation 1.2, we can solve for the force and it will be the max force transmission based on the max force occurring at time t=0.

$$\boldsymbol{F_{T_{Max}} = -mx''(0) = 2m\zeta V_0\omega_n} \quad (4.3)$$

<u>Critically Damped</u>

In the case of a critically damped system, we will have a damping ratio equal to 1, $\zeta = 1$. This is the damping situation that falls directly between the over and under damped systems (nonconciliatory motion and oscillations). We will see that this situation will occur when the roots of the characteristic equation are real double roots ($\lambda = -a/2$).

Again, we know through separation of variables a homogenous linear ODE of the form $x' + kx = 0$ has the general solution $x = Ce^{-rt}$ which is a good trial solution to use. To obtain the characteristic equation, we use the trial solution $x = Ce^{rt}$ and the first and second derivative x and plug it into our general equation

$$x'' = r^2e^{rt}$$

$$x' = re^{rt}$$

$$(r^2 + 2\omega_n r + \omega_n{}^2)e^{rt} = 0$$

$$\boldsymbol{r^2 + 2\omega_n r + \omega_n{}^2 = 0} \quad (4.4)$$

Now we find the roots of the characteristic equation which take the form $r_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b}, r_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b})$ from the quadratic equations. Since we have a double root, we know that the $a^2 - 4b$ is equal to zero so $r_{1,2} = -a/2$. In the case of two real roots, one solution of $x_1 = e^{-(r)t}$. For characteristic equation 4.4, the roots become

$$r = \frac{1}{2}\left(-2\omega_n \pm \sqrt{4\omega_n{}^2 - 4\omega_n{}^2}\right)$$

$$\boldsymbol{r_{1,2} = -\omega_n} \tag{4.5}$$

To form our basis, we need to find the second independent solution, x_2, we can set $x_2 = ux_1$. If we substitute this solution along with its first and second derivative into the general equation for linear homogenous equations, and collect terms we are left with $u''x_1 = 0$ and $u'' = 0$. Integrating we obtain $u = Ax + B$ and let A=1 and B=0 to get a second independent solution. This shows that when we have a double root of the characteristic equation, the basis of solutions equates to e^{-at} and te^{-at}. The general solution then becomes $x = (A + Bt)e^{-at}$. Using our characteristic equation and roots, the general solution becomes

$$\boldsymbol{x = A + Bte^{-\omega_n t}} \tag{4.6}$$

Using the first initial conditions of position zero at time zero, $x(0) = 0$, we can solve for A

$$x(0) = 0 = A + B(0)e^{-\omega_n(0)}$$

$$\boldsymbol{A = 0}$$

Plugging A into Eq. 4.6 yields

$$x = Bte^{-\omega_n t}$$

To solve for B, we can apply the other initial condition of at time zero we have the initial velocity, $x'(0) = V_0$

$$x' = B(e^{-\omega_n t} - \omega_n te^{-\omega_n t})$$

$$x'(0) = V_0 = B\left(e^{-\omega_n(0)} - \omega_n(0)e^{-\omega_n(0)}\right)$$

$$\boldsymbol{B = V_0}$$

Plugging in our coefficients back into Equation 4.6, we obtain the equation describing the displacement

$$\boldsymbol{x = V_0 te^{-\omega_n t}} \tag{4.7}$$

Solving for the velocity and acceleration based on the displacement

$$x' = V_0 e^{-\omega_n t} - V_0 te^{-\omega_n t}$$

$$\boldsymbol{x' = V_0 e^{-\omega_n t}(1 - \omega_n)} \tag{4.8}$$

$$x'' = [V_0 e^{-\omega_n t}(-\omega_n)] + [(1 - \omega_n t)(-\omega_n)V_0 e^{-\omega_n t}]$$

$$x'' = V_0 e^{-\omega_n t}[-\omega_n - \omega_n)(1 - \omega_n t)]$$

$$\boldsymbol{x'' = V_0 \omega_n e^{-\omega_n t}(\omega_n t - 2)} \tag{4.9}$$

Solving for the peak force, we know it occurs at the jerk or the third derivative of position times the mass or the derivative of the force with respect to time

$$\boldsymbol{F_{Max} = \frac{dF_T}{dt} = -mx''' = 0} \tag{5.0}$$

Where the third derivative of the position is

$$x''' = V_0\omega_n \frac{d}{dt}[(e^{-\omega_n t})(\omega_n t - 2)]$$

$$x''' = V_0\omega_n[-\omega_n(e^{-\omega_n t})(\omega_n t - 2) + e^{-\omega_n t}(\omega_n)]$$

$$\boldsymbol{x''' = V_0{\omega_n}^2 e^{-\omega_n t}(3 - \omega_n t)] = 0} \tag{5.1}$$

For this to be true, $e^{-\omega_n t}, V_0, \omega_n$ cannot equal zero. Then we can see that the maximum acceleration will be at time equals zero due to the negative in the exponent. Solving for the acceleration at time equals zero yields

$$x''(0) = V_0\omega_n e^{-\omega_n(0)}(\omega_n(0) - 2)$$

$$\boldsymbol{x''(0) = V_0\omega_n(-2)} \tag{5.2}$$

And plugging in Eq. 5.2 into the force equation

$$F_{T\,Max} = -mx''(0)$$

$$F_{T\,Max} = -m(V_0\omega_n(-2))$$

$$\boldsymbol{F_{T\,Max} = 2m(V_0\omega_n)} \tag{5.3}$$

<u>Underdamped Case</u>

Finally, we have the underdamped case where the damping ratio is less than one, $\zeta < 1$. In this case, we find the roots of the characteristic equation to be complex conjugates ($\lambda = \pm\frac{a}{2} + i\omega$) and will yield solutions to the ODE that are of complex nature.

Again, we know through separation of variables a homogenous linear ODE of the form $x' + kx = 0$ has the general solution $x = Ce^{-rt}$ which is a good trial solution to use. To obtain the characteristic equation, we use the trial solution $x = Ce^{rt}$ and the first and second derivative x and plug it into our general equation

$$x'' = r^2 e^{rt}$$

$$x' = re^{rt}$$

$$(r^2 + 2\zeta\omega_n r + {\omega_n}^2)e^{rt} = 0$$

$$\boldsymbol{r^2 + 2\zeta\omega_n r + {\omega_n}^2 = 0} \tag{5.4}$$

Now we find the roots of the characteristic equation which take the form $r_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b}, r_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b})$ from the quadratic equations. In the case of the complex conjugate, the radicand, $a^2 - 4b$, is negative and $4b - a^2$ is positive which is where we obtain the imaginary number, $\sqrt{-1} = i$. We can see that the radicand will transform below

$$\frac{1}{2}\sqrt{a^2-4b}$$

$$\frac{1}{2}\sqrt{-(4b-a^2)}$$

$$\sqrt{-(b-\frac{1}{4}a^2)}$$

$$\boldsymbol{i\sqrt{(b-\frac{1}{4}a^2)}} \quad (5.5)$$

To simplify the root, we set $\sqrt{(b-\frac{1}{4}a^2)}$ equal to ω and obtain $i\omega$ which is we can see is included in $\lambda = \pm\frac{a}{2} + i\omega$. With the case of complex conjugate roots, we can derive a complex exponential function, e^z, to generally find the real solutions where $z = r + it$. We must define our complex exponential function using real functions that include $e^r, \cos(t), \sin(t)$. Here we obtain the following

$$e^z = e^{r+it}$$

$$e^{r+it} = e^r e^{it}$$

$$\boldsymbol{e^r e^{it} = e^r(\cos(t) + i\sin(t))} \quad (5.6)$$

The transformation of e^{it} *to* $\cos(t) + i\sin(t)$ is derived from the Maclaurin series which yields the Euler Formula and when multiplied by e^r we obtain Equation 5.6. Continuing back to our roots, we can define r in Equation 5.6 as $r = -\frac{1}{2}at$ and $t = \omega t$.

$$\boldsymbol{e^{\lambda_1 t} = e^{\lambda_2 t} = e^{-\left(\frac{a}{2}\right)t + i\omega t} = e^{-\left(\frac{a}{2}t\right)}(\cos(\omega t) + i\sin(\omega t))} \quad (5.7)$$

If we now add $e^{\lambda_1 t}$ *and* $e^{\lambda_2 t}$, and multiply them by (1/2), then we obtain our first real solution

$$\boldsymbol{x_1 = e^{-\left(\frac{at}{2}\right)}\cos(\omega t)} \quad (5.8)$$

To obtain our second real solution, we can subtract $e^{\lambda_2 t}$ from $e^{\lambda_1 t}$ and multiply by $\frac{1}{2i}$.

$$\boldsymbol{x_2 = e^{-\left(\frac{at}{2}\right)}\sin(\omega t)} \quad (5.9)$$

In both Equation 5.8 and 5.9, as discussed after Equation 5.5 we recognize that $\omega = \sqrt{(b-\frac{1}{4}a^2)}$.

Observation of the quotient of the two solutions, x_1, x_2, being $\cot(\omega t)$ and that is not constant, we can say that the two solutions form a basis. Using the basis and general roots of the case, we obtain the general solution of the underdamped case.

$$\boldsymbol{x = e^{-\frac{a}{2}}\left(A\cos(\omega x) + B\sin(\omega x)\right)} \quad (6.0)$$

The roots of our characteristic equation (5.4) become

$$r = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n - 4(\omega_n)^2}}{2}$$

$$\boldsymbol{r_{1,2} = -\zeta\omega_n \pm i\sqrt{(\omega_n)^2 - (\zeta\omega_n)^2}} \tag{6.1}$$

For our problem we can derive the general equation using the overall approach and general equation 6.0 above of the underdamped case. The derivation of the general equation of this case is shown below

$$x = Ae^{r_1 t} + Be^{r_2 t}$$

$$x = Ae^{-\zeta\omega_n t + i\sqrt{(\omega_n)^2 - (\zeta\omega_n)^2 t}} + Be^{-\zeta\omega_n t - i\sqrt{(\omega_n)^2 - (\zeta\omega_n)^2 t}}$$

And applying to Eq. 6.0

$$x = e^{-\zeta\omega_n t}\left[Acos\left(\sqrt{(\omega_n)^2 - (\zeta\omega_n)^2 t}\right) + iAsin\left(\sqrt{(\omega_n)^2 - (\zeta\omega_n)^2 t}\right) + Bcos\left(\sqrt{(\omega_n)^2 - (\zeta\omega_n)^2 t}\right) - iBsin\left(\sqrt{(\omega_n)^2 - (\zeta\omega_n)^2 t}\right)\right]$$

$$x = e^{-\zeta\omega_n t}\left[(A+B)cos\left(\sqrt{(\omega_n)^2 - (\zeta\omega_n)^2 t}\right) + i(A-B)sin\left(\sqrt{(\omega_n)^2 - (\zeta\omega_n)^2 t}\right)\right]$$

It is noted from here that there are ways of representing amplitude and phase shifts of equations that take the form $Acos(x) + Bsin(x)$ by acknowledging that $\sqrt{A^2 + B^2} = C$ from the Pythagorean theorem in trigonometry and that there is a phase angle, Φ, where the $\tan(\phi) = B/A$ that we can simplify the formula from coefficients A and B to just C. We can also see from the triangle formed that $\sin(\phi) = \frac{A}{\sqrt{A^2+B^2}}$ and $\cos(\phi) = \frac{B}{\sqrt{A^2+B^2}}$.

$$x = e^{-\zeta\omega_n t}\sqrt{A^2 + B^2}\left[\left(\frac{A}{\sqrt{A^2 + B^2}}\right)cos\left(\sqrt{(\omega_n)^2 - (\zeta\omega_n)^2 t}\right) + \left(\frac{B}{\sqrt{A^2 + B^2}}\right)sin\left(\sqrt{(\omega_n)^2 - (\zeta\omega_n)^2 t}\right)\right]$$

$$x = e^{-\zeta\omega_n t}\sqrt{A^2 + B^2}\left[(\sin(\phi)cos\left(\sqrt{(\omega_n)^2 - (\zeta\omega_n)^2 t}\right) + \cos(\phi)sin\left(\sqrt{(\omega_n)^2 - (\zeta\omega_n)^2 t}\right)\right]$$

Applying that $\sqrt{A^2 + B^2} = C$ and simplifying, we obtain the general equation for displacement describing the underdamped case.

$$\boldsymbol{x = Ce^{-\zeta\omega_n t}sin\left(\sqrt{(\omega_n)^2 - (\zeta\omega_n)^2 t} + \phi\right)} \tag{6.2}$$

We again need to solve for the constant, C and phase angle by applying our initial conditions. First the position zero at time zero, $x(0) = 0$.

$$x(0) = 0 = Ce^{-\zeta\omega_n(0)}sin\left(\sqrt{(\omega_n)^2 - (\zeta\omega_n)^2(0)} + \phi\right) = Ce^{-\zeta\omega_n(0)}sin(0 + \phi)$$

$$\phi = 0 \; and \; C \neq 0$$

Taking the derivative of the position and applying our second initial condition of at time zero we have the initial velocity, $x'(0) = V_0$

$$x' = Ce^{-\zeta\omega_n t}\left[-\zeta\omega_n \sin\left(\sqrt{(\omega_n)^2 - (\zeta\omega_n)^2 t}\right) + \sqrt{(\omega_n)^2 - (\zeta\omega_n)^2}\cos\left(\sqrt{(\omega_n)^2 - (\zeta\omega_n)^2 t}\right)\right]$$

$$x'(0) = V_0 = Ce^{-\zeta\omega_n(0)}\left[-\zeta\omega_n\, sin\left(\sqrt{(\omega_n)^2-(\zeta\omega_n)^2(0)}\right)+\sqrt{(\omega_n)^2-(\zeta\omega_n)^2}\, cos\left(\sqrt{(\omega_n)^2-(\zeta\omega_n)^2(0)}\right)\right]$$

$$x'(0) = V_0 = Ce^{-\zeta\omega_n(0)}\left[\sqrt{(\omega_n)^2-(\zeta\omega_n)^2}\right]$$

$$C = \frac{V_0}{\left[\sqrt{(\omega_n)^2-(\zeta\omega_n)^2}\right]}$$

Now that we have C and Φ, we can plug them back into the general displacement equation to obtain the final displacement equation

$$\boldsymbol{x = \frac{V_0}{\left[\sqrt{(\omega_n)^2-(\zeta\omega_n)^2}\right]} e^{-\zeta\omega_n t} sin\left(\sqrt{(\omega_n)^2-(\zeta\omega_n)^2 t}\right)} \tag{6.3}$$

Now that we have the equation that describes the displacement of the undamped case from our general dynamic equation, we can take the derivative to obtain the velocity equation

$$x' = \frac{V_0}{\left[\sqrt{(\omega_n)^2-(\zeta\omega_n)^2}\right]} e^{-\zeta\omega_n t}\left[-\zeta\omega_n sin\left(\sqrt{(\omega_n)^2-(\zeta\omega_n)^2 t}\right)+\sqrt{(\omega_n)^2-(\zeta\omega_n)^2}\, cos\left(\sqrt{(\omega_n)^2-(\zeta\omega_n)^2(t)}\right)\right]$$

$$\boldsymbol{x' = \frac{V_0 e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}\left[-\zeta\omega_n\,\mathbf{sin}\left(\omega_n\sqrt{1-\zeta^2}t\right)+\left(\omega_n\sqrt{1-\zeta^2}\right) cos\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]} \tag{6.4}$$

Next, we can find the equation for the acceleration of the system by again taking the derivative, but this time of the velocity equation 6.4

$$x'' = \frac{V_0}{\omega_n\sqrt{1-\zeta^2}}\left[\left[(\zeta\omega_n)^2 e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)-(\zeta\omega_n)^2\left(\sqrt{1-\zeta^2}\right)e^{-\zeta\omega_n t}cos\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]\right.$$
$$\left.+\left[(-\zeta\omega_n)^2\left(\sqrt{1-\zeta^2}\right)e^{-\zeta\omega_n t}cos\left(\omega_n\sqrt{1-\zeta^2}t\right)-{\omega_n}^2(1-\zeta^2)e^{-\zeta\omega_n t}sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]\right]$$

If we factor out $(\zeta\omega_n)^2$ and $(1-\zeta^2)$ and simplify terms $(\zeta\omega_n)^2\left(\sqrt{1-\zeta^2}\right)e^{-\zeta\omega_n t}cos\left(\omega_n\sqrt{1-\zeta^2}t\right)$ and $(-\zeta\omega_n)^2\left(\sqrt{1-\zeta^2}\right)e^{-\zeta\omega_n t}cos\left(\omega_n\sqrt{1-\zeta^2}t\right)$ following the operators in the equation, then we can simplify the equation as shown below

$$x'' = \frac{V_0}{\omega_n\sqrt{1-\zeta^2}}\left[\left[e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\left((\zeta\omega_n)^2-\left(1-\zeta^2\right)\right)-2(\zeta{\omega_n}^2)\left(\sqrt{1-\zeta^2}\right)e^{-\zeta\omega_n t}cos\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]\right]$$

Simplifying further and distributing $\frac{V_0}{\omega_n\sqrt{1-\zeta^2}}$,

$$x'' = \frac{V_0}{\omega_n\sqrt{1-\zeta^2}}\left({\omega_n}^2(2\zeta-1)\right)\left(e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right)-\frac{V_0}{\omega_n\sqrt{1-\zeta^2}}\left[2(\zeta{\omega_n}^2)\left(\sqrt{1-\zeta^2}\right)e^{-\zeta\omega_n t}cos\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]$$

We can eliminate terms in red as shown below

$$x'' = \frac{V_0}{\cancel{\omega_n}\sqrt{1-\zeta^2}}\left(\omega_n^{\cancel{2}}(2\zeta-1)\right)\left(e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right)-\frac{V_0}{\cancel{\omega_n\sqrt{1-\zeta^2}}}\left[2(\zeta\omega_n^{\cancel{2}})\cancel{\left(\sqrt{1-\zeta^2}\right)}e^{-\zeta\omega_n t}cos\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]$$

Which then simplifies to the acceleration equation of the system

$$\boldsymbol{x'' = V_0\omega_n e^{-\zeta\omega_n t}\left[\frac{2\zeta^2-1}{\sqrt{1-\zeta^2}}\mathbf{sin}\left(\omega_n\sqrt{1-\zeta^2}t\right)-2\zeta\, cos\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]} \tag{6.5}$$

As in all the other cases, peak force occurs at the jerk or the third derivative of position times the mass or the derivative of the force with respect to time as in Equation 5.0. So, we must solve for the third derivative of position or the derivative of the acceleration as follows

$$x''' = V_0\omega_n\left[(-\zeta\omega_n)e^{-\zeta\omega_n t}\left[\frac{2\zeta^2-1}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right) - 2\zeta\, cos\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]\right]$$
$$+ V_0\omega_n\left[e^{-\zeta\omega_n t}\left[\frac{2\zeta^2-1}{\sqrt{1-\zeta^2}}\right]\omega_n\sqrt{1-\zeta^2}\cos\left(\omega_n\sqrt{1-\zeta^2}t\right) + 2\zeta\omega_n\sqrt{1-\zeta^2}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]$$

$$x''' = -V_0{\omega_n}^2\zeta e^{-\zeta\omega_n t}\left[\left[\frac{2\zeta^2-1}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right) - 2\zeta\, cos\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]\right]$$
$$+ V_0{\omega_n}^2\zeta e^{-\zeta\omega_n t}\left[(2\zeta^2-1)\omega_n\cos\left(\omega_n\sqrt{1-\zeta^2}t\right) + 2\zeta\omega_n\sqrt{1-\zeta^2}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]$$

Since in this case, $V_0e^{-\zeta\omega_n t}$, cannot equal zero, we can continue by dividing by $\cos\left(\omega_n\sqrt{1-\zeta^2}t\right)$ to be able to solve for time

$$\frac{2\zeta^2-1}{\sqrt{1-\zeta^2}}({\omega_n}^2\zeta)\frac{\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)}{cos\left(\omega_n\sqrt{1-\zeta^2}t\right)} + 2\zeta({\omega_n}^2\zeta) + (2\zeta^2-1){\omega_n}^2 + 2\zeta{\omega_n}^2\sqrt{1-\zeta^2}\frac{\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)}{cos\left(\omega_n\sqrt{1-\zeta^2}t\right)} = 0$$

We know that $\frac{sin}{cos} = tan$, so we can simplify

$$\frac{2\zeta^2-1}{\sqrt{1-\zeta^2}}({\omega_n}^2\zeta)tan\left(\omega_n\sqrt{1-\zeta^2}t\right) + 2\zeta({\omega_n}^2\zeta) + (2\zeta^2-1){\omega_n}^2 + 2\zeta{\omega_n}^2\sqrt{1-\zeta^2}tan\left(\omega_n\sqrt{1-\zeta^2}t\right) = 0$$

Moving the blue terms to the other side of the equation, factoring out the tan and simplifying yields

$$-\left[-\frac{2\zeta^2-1}{\sqrt{1-\zeta^2}}({\omega_n}^2\zeta) + \frac{\left(2\zeta{\omega_n}^2(1-\zeta^2)\right)}{\sqrt{1-\zeta^2}}\right]tan\left(\omega_n\sqrt{1-\zeta^2}t\right) = 2\zeta^2{\omega_n}^2 + (2\zeta^2-1){\omega_n}^2$$

Simplifying the terms in the brackets

$$-\frac{-({\omega_n}^2\zeta)(2\zeta^2-1) + {\omega_n}^2\left(2\zeta(1-\zeta^2)\right)}{\sqrt{1-\zeta^2}}tan\left(\omega_n\sqrt{1-\zeta^2}t\right) = 2\zeta^2{\omega_n}^2 + (2\zeta^2-1){\omega_n}^2$$

Moving the terms in blue to the other side of the equation

$$tan\left(\omega_n\sqrt{1-\zeta^2}t\right) = \frac{({\omega_n}^2\zeta)(4\zeta^2-1)\sqrt{1-\zeta^2}}{({\omega_n}^2\zeta)(2\zeta^2-1) - {\omega_n}^2\left(2\zeta(1-\zeta^2)\right)}$$

$$tan\left(\omega_n\sqrt{1-\zeta^2}t\right) = \frac{(4\zeta^2-1)\sqrt{1-\zeta^2}}{4\zeta^3-3\zeta}$$

Now we must solve for time, t. To do this, we have to move everything to the right side of the equation. This will give the arc tan or $\tan^{-1}(\,)$.

$$\boldsymbol{t = \left(\frac{1}{\omega_n\sqrt{1-\zeta^2}}\right)\tan^{-1}\left(\frac{(4\zeta^2-1)\sqrt{1-\zeta^2}}{4\zeta^3-3\zeta}\right)} \tag{6.6}$$

Assessing Eq. 6.6, we can graph the time versus the damping ratio as seen in Figure 3 below

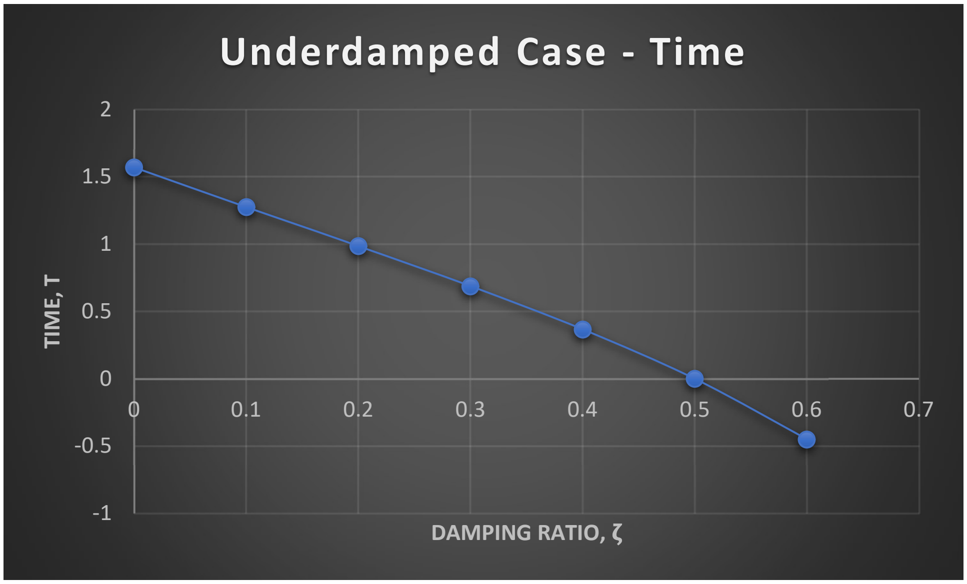

Figure 3: Underdamped Case – Time vs Damping Ratio

We see that at time is equal to zero, the damping coefficient is 0.5. We can see that solving for ζ that when $4\zeta^2 = 1$, the term in the brackets goes to zero and we can solve for $\zeta = \frac{1}{2}$. From the graph, we can see that when time is greater than zero, the damping ratio is between 0 and ½. When the damping ratio is above ½ up to 1 (underdamped case $\zeta < 1$) then the time will be zero. This means that the force transmitted will differ depending on the damping ratio. The max force transmitted when the damping ratio is between zero and one half, $0 < \zeta < \frac{1}{2}$, is shown below where the time is equal to Equation 6.6, $t = \left(\frac{1}{\omega_n\sqrt{1-\zeta^2}}\right)\tan^{-1}\left(\frac{(4\zeta^2-1)\sqrt{1-\zeta^2}}{4\zeta^3-3\zeta}\right)$

$$\boldsymbol{F_{T_{Max}} = -mx'' = -mV_0\omega_n e^{-\zeta\omega_n t}\left[\frac{2\zeta^2-1}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right) - 2\zeta\cos\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]} \tag{6.7}$$

The max transmitted force when the damping ration is between ½ and 1, $\frac{1}{2} < \zeta < 1$, is when time is equal to zero, t=0

$$F_{T_{Max}} = -mx'' = -mV_0\omega_n e^{-\zeta\omega_n(0)}\left[\frac{2\zeta^2-1}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}(0)\right) - 2\zeta\cos\left(\omega_n\sqrt{1-\zeta^2}(0)\right)\right]$$

$$F_{T_{Max}} = -mV_0\omega_n e^{-\zeta\omega_n(0)}\left[\frac{2\zeta^2-1}{\sqrt{1-\zeta^2}}\sin(0) - 2\zeta\cos(0)\right]$$

$$F_{T_{Max}} = 2mV_0\omega_n\zeta \quad (6.8)$$

Case Summary

The below tables will summarize the cases analyzed above

Case	Damping Ratio	Position
Undamped	0	$x = \frac{V_0}{\omega_n} sin\omega_n t$
Overdamped	>1	$x = \frac{V_0}{2\omega_n\sqrt{\zeta^2-1}}\left(e^{\left(-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)t} - e^{\left(-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)t}\right)$
Critically Damped	=1	$x = V_0 te^{-\omega_n t}$
Underdamped	<1	$x = \frac{V_0}{\left[\sqrt{(\omega_n)^2-(\zeta\omega_n)^2}\right]} e^{-\zeta\omega_n t} sin\left(\sqrt{(\omega_n)^2-(\zeta\omega_n)^2 t}\right)$

Table 1: Summary of Position

Case	Damping Ratio	Acceleration
Undamped	0	$x'' = -V_o\omega_n sin\omega_n t$
Overdamped	>1	$x'' = \frac{V_0\omega_n}{2\sqrt{\zeta^2-1}}\left[\left(-\zeta+\sqrt{\zeta^2-1}\right)^2 e^{\left(-\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)t} - \left(-\zeta-\sqrt{\zeta^2-1}\right)^2 e^{\left(-\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)t}\right]$
Critically Damped	=1	$x'' = V_0\omega_n e^{-\omega_n t}(\omega_n t - 2)$
Underdamped	<1	$x'' = V_0\omega_n e^{-\zeta\omega_n t}\left[\frac{2\zeta^2-1}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right) - 2\zeta\, cos\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]$

Table 2: Summary of Acceleration

Case	Damping Ratio	Transmitted Force
Undamped	0	$F_{T_{Max}} = mV_o\omega_n$
Overdamped	>1	$F_{T_{Max}} = 2m\zeta V_0\omega_n$
Critically Damped	=1	$F_{T_{Max}} = 2m(V_0\omega_n)$
Underdamped	0 – ½	$F_{T_{Max}} = -mV_o\omega_n e^{-\zeta\omega_n t}\left[\frac{2\zeta^2-1}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right) - 2\zeta\, cos\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]$
	½ - 1	$F_{T_{Max}} = 2mV_0\omega_n\zeta$

Table 3: Summary of Transmitted Force

Ratio of Force Transmission

To analyze the ratio of force transmission we will compare the transmitted force of each case with the force transmitted in the undamped case.

1. $\frac{Undamped}{Undamped}$

$$R_{undamped} = \frac{mV_o\omega_n}{mV_o\omega_n} = 1 \quad (6.9)$$

2. $\frac{Overdamped}{Undamped}$

$$\boldsymbol{R_{Overdamped}} = \frac{\boldsymbol{2m\zeta V_0 \omega_n}}{\boldsymbol{mV_o\omega_n}} = \boldsymbol{2\zeta} \tag{7.0}$$

3. $\frac{Critcally\ damped}{Undamped}$

$$\boldsymbol{R_{Critically\ damped}} = \frac{\boldsymbol{2m(V_0\omega_n)}}{\boldsymbol{mV_o\omega_n}} = \boldsymbol{2} \tag{7.1}$$

4. $\frac{Underdamped}{Undamped}$
   a. $0 < \zeta < \frac{1}{2}$
      i. $t = \left(\frac{1}{\omega_n\sqrt{1-\zeta^2}}\right)\tan^{-1}\left(\frac{(4\zeta^2-1)\sqrt{1-\zeta^2}}{4\zeta^3-3\zeta}\right)$

$$\boldsymbol{R_{underdamped}} = \frac{-\boldsymbol{mV_0\omega_n e^{-\zeta\omega_n t}}\left[\frac{\boldsymbol{2\zeta^2-1}}{\sqrt{\boldsymbol{1-\zeta^2}}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right) - 2\zeta\cos\left(\omega_n\sqrt{1-\zeta^2}t\right)\right]}{\boldsymbol{mV_o\omega_n}}$$

$$\boldsymbol{R_{underdamped}} = e^{-\zeta\omega_n t}\left[\frac{2\zeta^2-1}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right) - 2\zeta\cos\left(\omega_n\sqrt{1-\zeta^2}t\right)\right] \tag{7.2}$$

   b. $\frac{1}{2} < \zeta < 1$

$$\boldsymbol{R_{underdamped}} = \frac{\boldsymbol{2mV_0\omega_n\zeta}}{\boldsymbol{mV_o\omega_n}} = \boldsymbol{2\zeta} \tag{7.3}$$

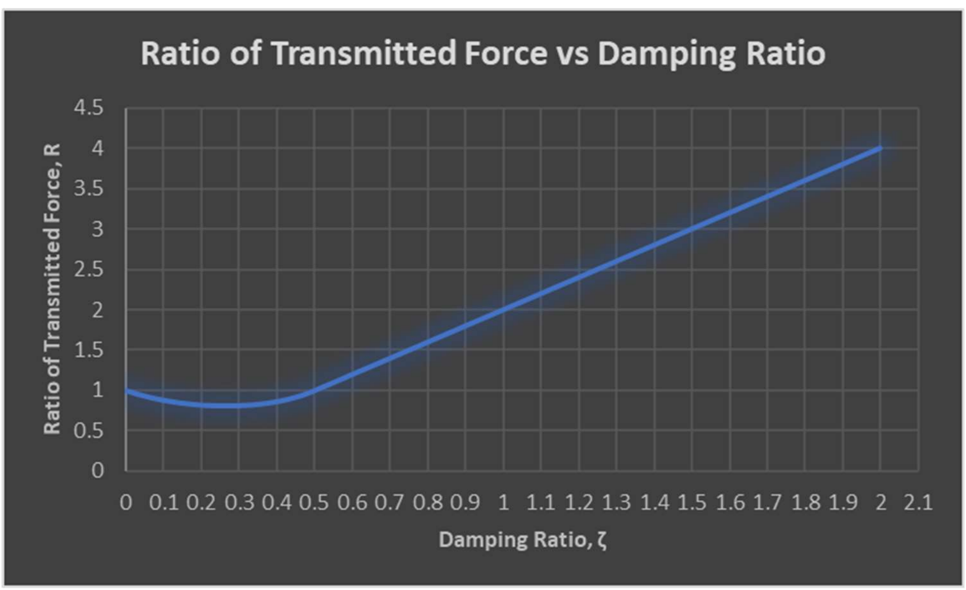

Figure 4: Plot of Ratio of Transmitted Force vs Damping Ratio

The optimum damping ratio (minimizes the peak force transmitted to the floor) can be found at the minimum point of the curve which according to the graph and calculations of the ratio of transmitted force occurs at a damping ratio of $\zeta = 0.26$ which yields a ratio of transmitted force of, $R = 0.810188$. With the undamped case having a ratio of transmitted force of $R = 1$, then the percent difference between the undamped case and the case of optimum damping (minimum peak force transmitted) is

$$Percent\ Decrease = (1 - .810188) * 100$$

$$\boldsymbol{Percent\ Decrease = 18.98\%} \tag{7.4}$$

Now that we have the optimum damping ratio, we can determine the damping coefficient as a function of the mass and the spring constant of the pad using Equation 1.6 ($\zeta = \frac{c}{c_c}$), 1.9 ($c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n$), and 2.0 ($\frac{c}{m} = 2\zeta\omega_n$).

$$c = 2\zeta m\omega_n$$

$$c = 2(0.26)m\omega_n$$

$$\boldsymbol{c = 0.52m\omega_n} \tag{7.5}$$

Discussion

The mathematical methods used in this report prove to be complex but form a basis of how many mechanical problems can be solved. Homogeneous linear ordinary differential equations with constant coefficients prove to be vital in many mechanical problems such as mass spring systems. This problem lends itself well as we can view the system as a mass spring system with a dashpot (damped system). This type of equation takes the form of y''+ay'+by=0, which we see in the dynamic governing equation of the system shown in Equation 1.5.

Steps involved in solving the ODE for differing cases involves finding trial solutions, obtaining the characteristic equation, finding the roots of the characteristic equation, finding the basis of the ODE, and finally obtaining the expression of the general solution. Applying the initial conditions of the problem, we can solve for the constant coefficients and we find the expression for the position. The expressions will differ based on the type of root that is derived from the characteristic equation. It can be either distinct and real, a real double root, or a complex conjugate. The different roots represent different cases that all relate to the motion of mechanical systems or the flow of current in an electrical system. Since, in our case we are looking for the force transmission of each case, and we know from Newton's second law that the force is equal to mass times the acceleration, we can derive the acceleration from the position by taking the derivative of position twice.

A system with no damping, would theoretically oscillate forever. Since all systems have at least a little bit of damping by nature, we can view an undamped system by taking a picture of the system motion for a small amount of time. For a mass spring system, we can view the harmonic oscillations that are essentially the outside forces that restore the system to its equilibrium point after being displaced. The harmonic oscillation has a constant amplitude and a sinusoidal motion associated. With a damped system we can see different changes that may occur. When the system is overdamped, we have distinct real roots for the characteristic equation. From the case name we can see that this type of damping removes energy in the system at a high rate from the system which does not allow oscillations and steady state is achieved slowly. This case is also known as a non-oscillatory state. An underdamped system yields complex conjugate roots and the amplitude of the oscillations start out large and steadily decline until they reach zero. In the critically damped system, the system quickly returns to equilibrium or steady state without any oscillations, which can be viewed as a damping case in between the under and overdamped cases. In this case, the initial velocity can cause overshoot if a non-zero value.

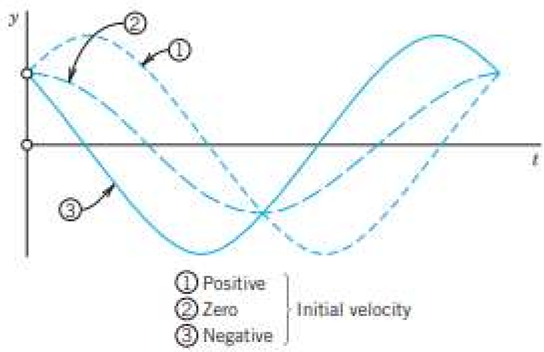

Figure 5: Harmonic Oscillation

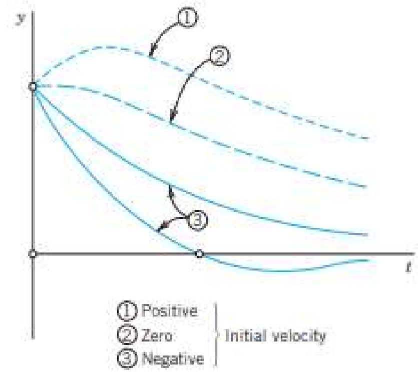

Figure 6: Critical Damping

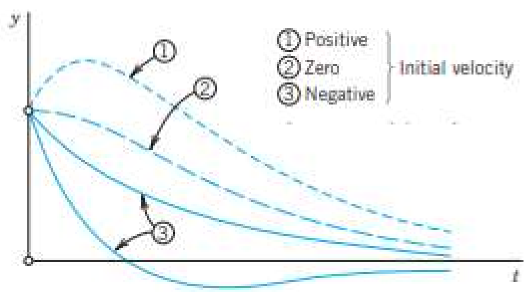

Figure 7: Overdamping

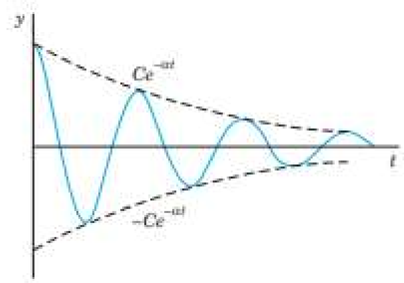

Figure 8: Underdamping

With all of the different potential damping ratios, it is difficult to decide which would be the most efficient to choose for a system to minimize the force to the floor. This is best shown in Figure 4. Where the optimum damping ratio is at the minimum point of the curve with a damping ratio of 0.26. This should be taken as an approximation as the actual damping ratio could be more accurately depicted if taken out to more decimals. With an undamped system, the ratio of force transmission is 100%. When looking at a large damping ratio, we can see that the ratio of force transmission increases linearly at a rapid rate. This would prevent oscillations but would not be optimal for force transmission to the floor. When we use a damping ratio of one for the critically damped system, we see the ratio of force transmission is doubled from the undamped system, which is a smaller amount of force transmission than the overdamped system, but still not damped enough. The underdamped system force transmission depends on the damping ratio used since as seen in Figure 3, the damping ratio above 0.5 occurs at a time above zero. At this point in Figure 4, the graph is already increasing. The optimal damping ratio yields the lowest ratio of force transmission, which is an underdamped system. The percent decrease of the force of the undamped system is approximately 20%, which means that the force transmission is minimized with the damping ratio of 0.26. Finding the damping coefficient of the system is important as the expression found can be used for many different initial velocities of the hammer and the mass.

Ultimately, we were able to find the most efficient system with the optimal damping ratio. Future work could include looking into other methods to minimize the force applied to the floor. There are ways to use intentional vibrations to the system that in a way, cancel out or disrupt the vibrations created by the system. For a simple system, a damping pad with the 0.26 damping ratio (or damping coefficient of $0.52m\omega_n$) may be sufficient, but if there was a desire to reduce the force even more, the vibration isolations may be useful.

References:

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