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Heating of a Motor Cable Junction

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## List of Symbols

$\alpha$	Thermal Diffusivity
A	Arbitrary Constant
Al	Aluminum
B	Arbitrary Constant
C	Arbitrary Constant
Cu	Copper
e	Exponential
$\Sigma$	Series Sum
f	Function
i	Imaginary Number
k	Separation Constant
n	Series Component
$\omega$	Omega
P	Period
$\phi$	Phi
T	Temperature
t	Time
v	Velocity
x	Position

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## Abstract

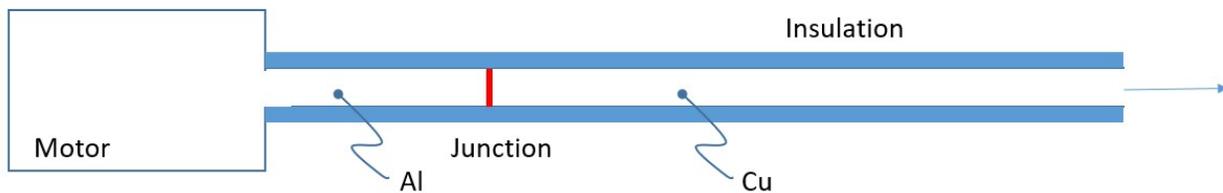
During the operation of a motor, extra heat can be produced in a few situations where the motor will draw several times its rated current. The motor has an aluminum motor stator that is connected to a copper power cable by a welded or brazed joint. At the joint, the high electrical resistance causes extra heat. When the motor draws more current than it is rated for, the heat at the junction increases a larger non-conducting interface grows. Two situations are studied where the initial condition at the motor is either a sine wave or a square wave. By utilizing a partial differential equation in the form of the heat diffusion equation and Fourier series analysis, we are able to show how the temperature distribution varies as we move the junction further from the motor which decreases as we move further from the source even with differing inputs. We also discover that the wave is not preserved as it moves down the cable, which aids in the search for the optimal location of the junction. With Figures 3 and 5 through 12, we can present the ways to avoid the metallurgical phenomenon altogether and explain the reasoning to support our findings for the small Motor-Generator manufacturer as the material science engineers study the phenomenon. With our findings, it will be up to the manufacturer how far they are able to move the junction away from the motor to reduce the non-conducting interface but stay within their design boundaries of practical lengths of cable.

## Introduction/Assumptions

We look to study a metallurgical phenomenon for a small Motor-Generator manufacturer. There is a potential source of a problem at a brazed or welded joint between an aluminum motor stator winding and one of its copper power-cables. As the cable is heated, an electrical non-conducting interface can sometimes form.

During operation, extra heat from the motor could be produced by stalling the motor so that it draws several times its rated current, or by running the motor with a mechanical overload. Since the electrical resistance at the junction is high, the current flowing through it produces a high level of electrical heating in the form of  $(I^2R)$  loss as additional heating. This makes the non-conducting interface grow larger. While the metallurgical phenomena is being studied by the material science engineers, our consulting company has been asked to study ways of preventing the occurrence altogether.

One possible solution to be studied is to move the joint further away from the motor, so that during a short time, it does not become as hot as the motor. To investigate the value of this potential solution, we can idealize the motor and connection as shown in Figure 1.



*Figure 1: Motor with Brazed/Welded Junction*

The cable is assumed to be infinitely long and insulated so well that it loses no heat by conduction through the insulation. Since we have differing materials, we know that the wire is non-homogeneous. We also assume that the aluminum and copper portions of the cable have identical cross-sections. For our first approximation, we assume that the thermal diffusivity of the aluminum is that same as the thermal diffusivity of the copper at  $4.353 \text{ ft}^2/\text{hr}$ . In the problem at hand, we will also assume that the heat flow is in one direction (one dimensional) and that the thermal conductivity is constant. This will be a transient problem, not steady state. The weld/braze joint is not considered as a separate filler metal or material.

In conducting the investigation, a few different situations and questions must be answered; namely:

1.) If the temperature of the motor end of aluminum cable is a sine wave,  $T(0, t) = T_0 \sin \omega_0 t$ , with period,  $P = \frac{2\pi}{\omega_0} = 1 \text{ hr}$ , how does the peak junction temperature, expressed as a fraction of  $T_0$ , vary with the position of the junction?

2.) If  $T(0, t)$  is a square wave, as shown in Figure 2, how does the temperature of the junction vary with time if the junction is located one foot from the motor?

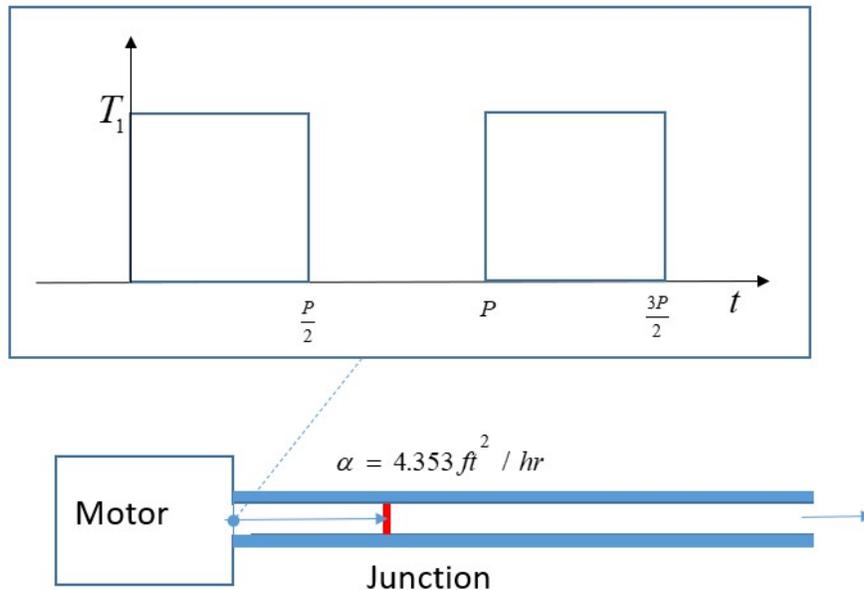


Figure 2: Temperature of the Motor as a Square Wave

3.) Is the waveform of the temperature wave preserved as it moves along the cable? If not, why? To help the explanation, the velocities of the first five components of the wave should be tabulated.

## Conclusions

The purpose of this report is to analyze differing inputs at the motor to evaluate how we can prevent the excess current and resulting temperature at the junction of the motor stator and the copper cable from creating a large non-conducting interface. To analyze and present our results, we study two differing cases; one where the temperature of the motor end of the aluminum cable is a sine wave,  $T(0, t) = T_0 \sin \omega_0 t$ , with period,  $P = \frac{2\pi}{\omega_0} = 1hr$  and the other when the motor end of the aluminum cable is a square wave given in Figure 2.

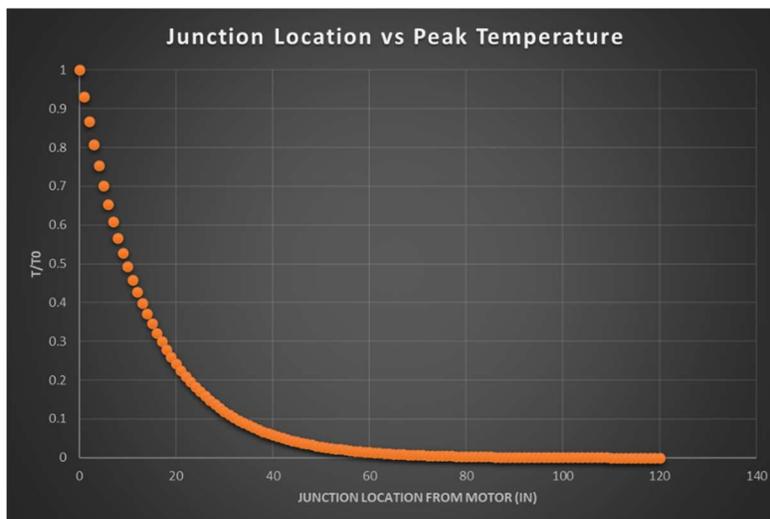
Starting with the heat diffusion equation, we use separation of variables method to derive two ordinary differential equations where we could apply superposition principals and the Fourier series to obtain our final distribution for this transient case.

$$T(x, t) = \sum_{n=0}^{\infty} \{e^{-\phi_n x} [B_n \cos (2\phi_n^2 at - \phi_n x) + C_n \sin (2\phi_n^2 at - \phi_n x)]\}$$

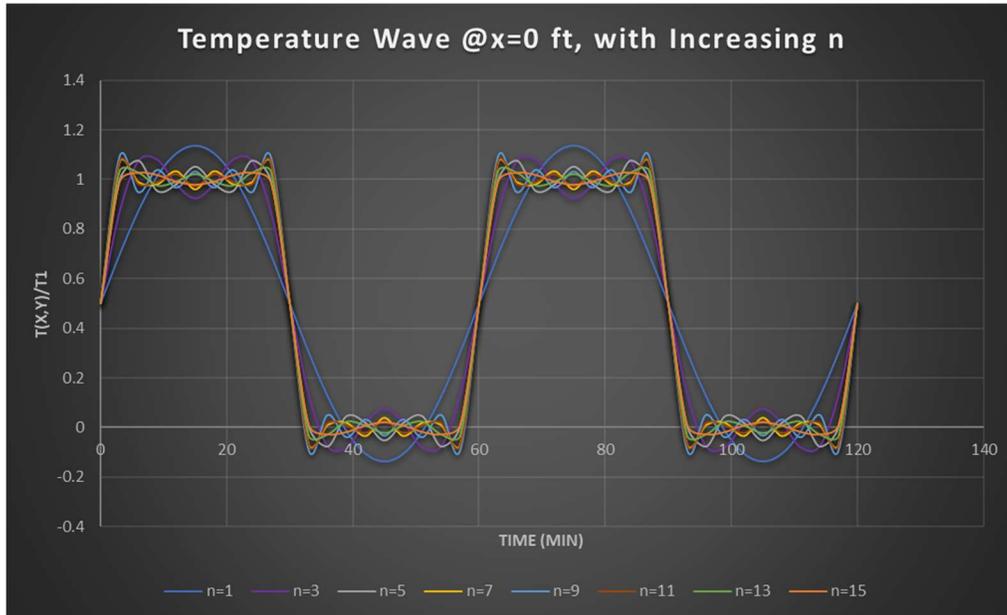
We then used the initial conditions to find the values of the coefficients. Our first situation, we are interested in the how the peak junction temperature, expressed as a fraction of  $T_0$  varies with the position of the junction. When applying  $T(0, t) = T_0 \sin \omega_0 t$ , we obtain the distribution. Since we are looking for the peak junction temperature, we know that the this will occur when the sine wave is equal to 1, so we then have

$$\frac{T(x, t)}{T_0} = e^{-\sqrt{\frac{\omega_0}{2\alpha}} x} \sin \left( \omega_0 t - \sqrt{\frac{\omega_0}{2\alpha}} x \right), \text{ At Max Temp } \frac{T(x, t)}{T_0} = e^{-\sqrt{\frac{\omega_0}{2\alpha}} x}$$

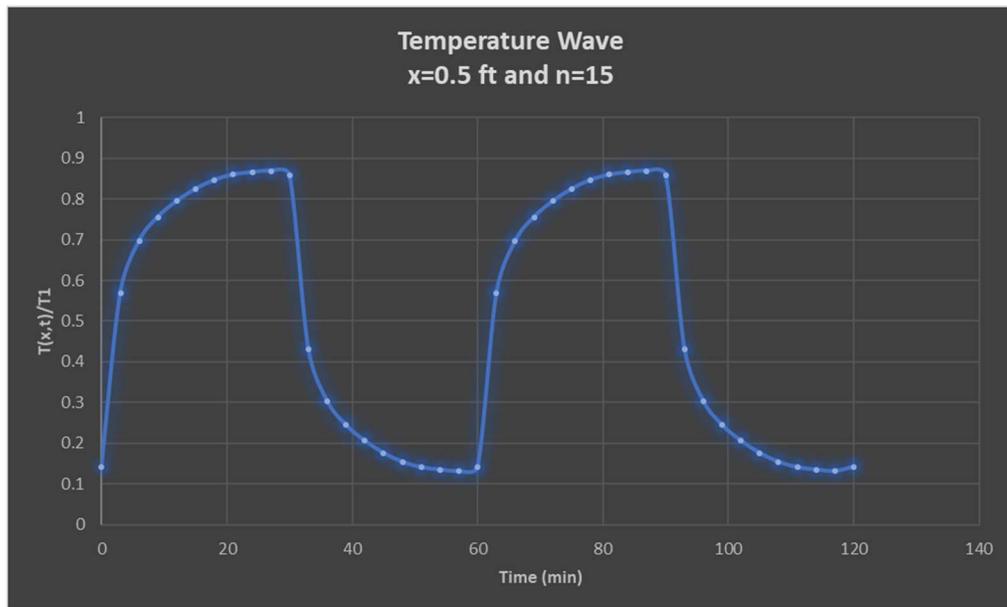
The peak temperature varies with time as shown in Figure 3, which approaches zero as we reach infinity, in our case stopping at 120 inches.



When evaluating the square wave given, we can apply the Fourier series analysis to find its form and again set it equal to our final distribution. In this case, we want to know the how the temperature of the junction varies with time if the junction is located one foot from the motor. We can see using the Fourier Series, we are close to approaching the square wave at the location of zero when we use an n of 15, so we know our Fourier calculation is valid.

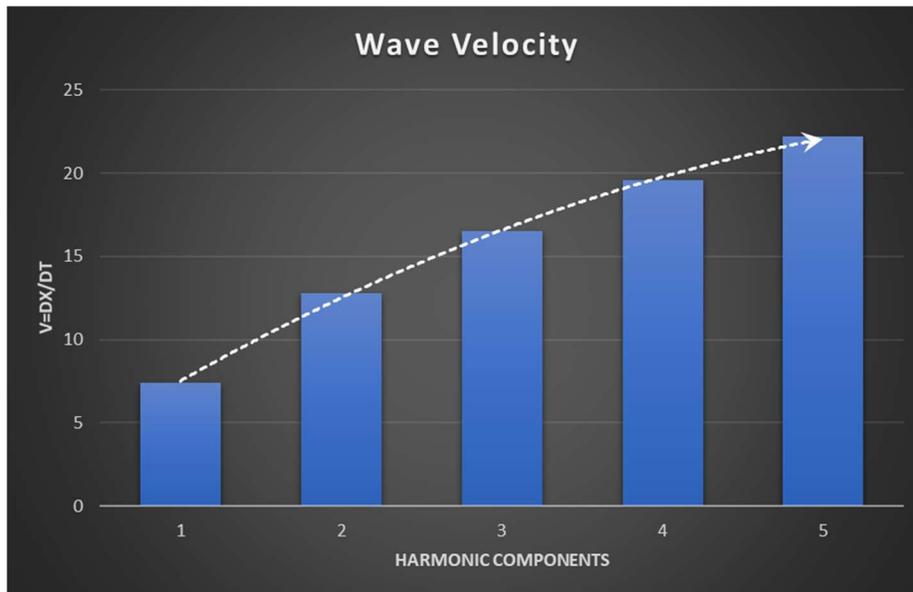
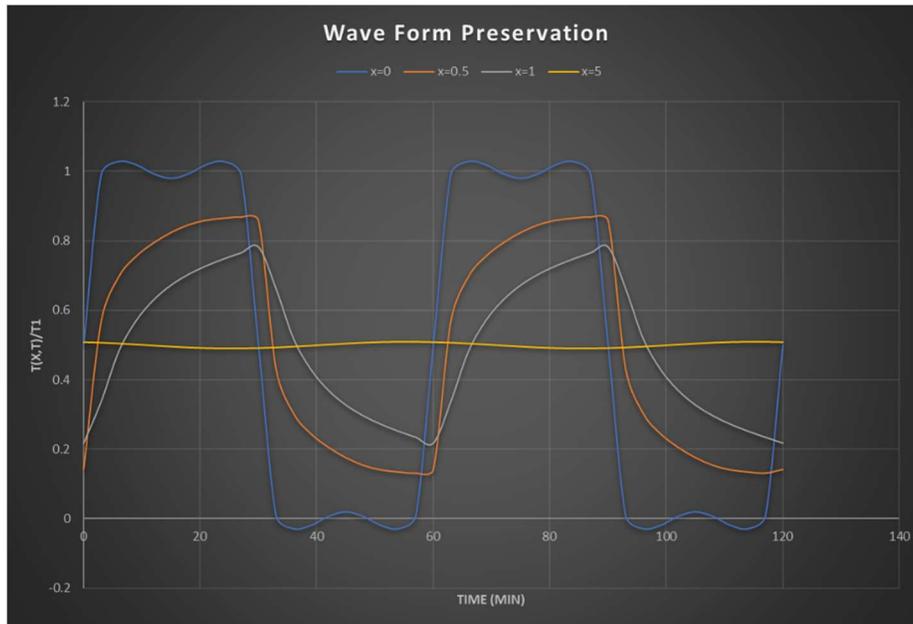


We can view the temperature wave when the junction is one foot from the motor using an approximation of the series at n=15.



With this square wave, we can also evaluate how the temperature wave is preserved as it moves down the cable. We can see this in two ways; plotting the temperature wave as it moves further

away from the motor, and also by tabulating the velocity of the first few components of the wave. We can see that it is not preserved and dissipates as we move down the cable.



All of this information can be used to decide as to the location of the junction to prevent the non-conducting interface growing no matter what the motor input. The presentation of this information will allow the manufacturer to decide how to prevent this metallurgical phenomenon while remaining in their design envelope. We could recommend the location of the junction, but this will have to be determined after deliberation from the manufacturer. All we can do is supply the information to allow them to make the best decision.

## Analysis

To start the analysis, we want to calculate the distribution of the junction temperature where  $T(0, t) = T_0 \sin \omega_0 t$  and  $P = \frac{2\pi}{\omega_0} = 1 \text{ hr}$ . To analyze this, we must use the one-dimensional heat diffusion equation where  $\alpha$  is the thermal diffusivity

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \quad (1.0)$$

Rearranging equation 1.0, we have

$$\nabla^2 T = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (1.1)$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (1.2)$$

To solve this partial differential equation, we use separation of variables method where we first substitute the product,  $T = X(x)Y(t)$ , into equation 1.2

$$X''(x)Y(t) = \frac{1}{\alpha} X(x)Y'(t) \quad (1.3)$$

To separate the variables, we divide both sides by  $\frac{1}{\alpha} X(x)Y(t)$ .

$$\frac{X''(x)Y(t)}{\frac{1}{\alpha} X(x)Y(t)} = \frac{\frac{1}{\alpha} X(x)Y'(t)}{\frac{1}{\alpha} X(x)Y(t)}$$

$$\frac{X''(x)a}{X(x)} = \frac{Y'(t)}{Y(t)}$$

$$\frac{X''(x)}{X(x)} = \frac{1}{a} \frac{Y'(t)}{Y(t)} \quad (1.4)$$

Now the variables are separated so that the left side of the equation only depends on  $x$  and the right side of the equation only depends on  $t$ . We see that both side of the equation must be constant because if they were variable, then any changes to the variables would affect only one side of the equation. For this reason, we will set equation 1.4 equal to a separation constant,  $k^2$ .

$$\frac{X''(x)}{X(x)} = \frac{1}{a} \frac{Y'(t)}{Y(t)} = k^2 \quad (1.5)$$

Multiplying the denominator on both sides and moving to the other side of the equation yields two ordinary differential equations that stemmed from the original partial differential equation

$$X'' - k^2X = 0 \quad (1.6)$$

$$Y' - \alpha k^2Y = 0 \quad (1.7)$$

We recall that equation 1.6 has two distinct real roots and the general solution is

$$X(x) = Ae^{kx} + Be^{-kx} \quad (1.8)$$

The general solution of equation 1.7 is defined as

$$Y(t) = Ce^{\alpha k^2 t} \quad (1.9)$$

If the separation constant is equal to zero, we find that the general solutions would take the form of  $X(x) = A_0x + B_0$  and  $Y(t) = C_0$ . These will be eliminated when we have our steady state portion. Since the temperature distribution was defined as the product,  $T = X(x)Y(y)$ , the temperature distribution is the product of equation 1.8 and 1.9. To look at the temperature distribution of the junction, we use the Fourier series to represent the distribution of the junction temperature as

$$T(x, t) = A_0x + B_0 + \sum_{n=1}^{\infty} e^{k^2 \alpha t} (A_n e^{kx} + B_n e^{-kx}) \quad (2.0)$$

We see that we eliminate  $A_0x + B_0$  since we are concerned with the transient portion of the equation. We were given that  $T(0, t) = T_0 \sin \omega_0 t$  and  $P = \frac{2\pi}{\omega_0} = 1hr$ , so we need to evaluate the temperature at position zero. Using equation 2.0 to evaluate at position 0, we show that  $T(0, t) = (A_0 + B_0)e^{k^2 \alpha t}$ , but we also are given  $T(0, t) = T_0 \sin \omega_0 t$ . With this we can say that we define  $\omega_n = \omega_0 n = \frac{2\pi}{P} n$ .

Since equation 2.0 is in complex form shown by the use of exponentials, we must produce  $T(0, t)$  in complex form as well. We use the definitions

$$\cos(x) = \frac{e^{xi} + e^{-xi}}{2} \quad \text{and} \quad \sin(x) = \frac{e^{xi} - e^{-xi}}{2i}$$

The normal procedure is to convert and then solve for the coefficients. For example, an equation taking the form  $f(t) = d + \sum_{n=1}^{\infty} [A_n \cos(nt) + b_n \sin(nt)]$  using the definitions would show

$$f(t) = d + \sum_{n=1}^{\infty} \left[ A_n \left( \frac{e^{int} + e^{-int}}{2} \right) + B_n \left( \frac{e^{int} - e^{-int}}{2i} \right) \right]$$

Since we know that  $\frac{1}{i} = -i$  then

$$f(t) = d + \sum_{n=1}^{\infty} \frac{A_n - iB_n}{2} e^{int} + \sum_{n=1}^{\infty} \frac{A_n + iB_n}{2} e^{-int}$$

$$= \sum_{n=-\infty}^{\infty} C_n e^{int}, \text{ where } C_n = \begin{cases} d & n = 0 \\ \frac{A_n - iB_n}{2} & n = 1, 2, 3 \dots \\ \frac{(A_{-n} + iB_{-n})}{2} & n = -1, -2, -3 \dots \end{cases}$$

We also define  $A_n = \frac{1}{p} \int_{-p}^p \cos(nt) f(t) dt$ ,  $B_n = \frac{1}{p} \int_{-p}^p \sin(nt) f(t) dt$ ,  $d = \frac{1}{2p} \int_{-p}^p f(t) dt$ . So, for positive  $n$ ,  $C_n = \frac{1}{2}(A_n - iB_n) = \frac{1}{p} \int_{-p}^p [\cos(nt) - i\sin(nt)] f(t) dt = \frac{1}{p} \int_{-p}^p [e^{-int} f(t) dt]$ .

If we were looking at  $f(t) = \sin(t)$  then  $C_n = \frac{1}{2p} \left[ \frac{(e^{inp} - e^{-inp})}{n^2 - 1} \right]$  and  $c_1 = \frac{1}{2i}$ .

We can evaluate our situation in the same way using the converted  $\omega_n$ . Here we can rewrite  $T(0, t) = T_0 \sin\left(\frac{2\pi}{p} nt\right)$ . In this way, we can say that

$$\sin\left(\frac{2\pi}{p} nt\right) = \frac{e^{\frac{2\pi}{p} nti} - e^{-\frac{2\pi}{p} nti}}{2i}$$

$$= T_0 \left( \frac{e^{\frac{i2\pi}{p} nt} - e^{-\frac{i2\pi}{p} nt}}{2i} \right)$$

Since we know that  $\frac{1}{i} = -i$  then

$$= -iT_0 \left( \frac{e^{\frac{i2\pi}{p} nt} - e^{-\frac{i2\pi}{p} nt}}{2} \right)$$

$$= iT_0 \left( \frac{e^{\frac{i2\pi}{p} nt} + e^{-\frac{i2\pi}{p} nt}}{2} \right)$$

$$= \frac{1}{2} iT_0 \left( e^{\frac{i2\pi}{p} nt} + e^{-\frac{i2\pi}{p} nt} \right)$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} C_n e^{i\omega_n t}$$

Which leads to the complex Fourier series representation

$$T(0, t) = T_0 \sin \omega_n t = \frac{1}{2} \sum_{n=1}^{\infty} C_n e^{i\omega_n t} \quad (2.1)$$

Using equation 2.1 and remembering that  $T(0, t) = (A_0 + B_0)e^{k^2 \alpha t}$ , we can state

$$(A_0 + B_0)e^{k^2 \alpha t} = \frac{1}{2} \sum_{n=1}^{\infty} C_n e^{i\omega_n t} \quad (2.2)$$

Now we want to solve for  $k$  in the equation. Since the bases for both exponentials are the same, then the two expressions are only equal if the exponents are also equal, where we say

$$k^2 \alpha t = i\omega_n t$$

Eliminating the time variable and using the definition,  $\omega_n = \omega_0 n$

$$k^2 \alpha = i\omega_0 n$$

$$k = \pm \sqrt{\frac{i\omega_0 n}{\alpha}} \quad (2.3)$$

As an aside, we must recognize some complex definitions as follows

$$i = e^{i\left(\frac{\pi}{2}\right)} = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) = 1$$

$$\sqrt{i} = i^{\frac{1}{2}} = e^{\left(i\left(\frac{\pi}{2}\right)\right)^{\frac{1}{2}}} = e^{i\left(\frac{\pi}{4}\right)} = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} = \frac{1+i}{\sqrt{2}}$$

Due to these complex definitions, then we can refine  $k$  further as shown below

$$k = \pm \frac{1+i}{\sqrt{2}} \sqrt{\frac{\omega_0 n}{\alpha}} = \pm (1+i) \sqrt{\frac{\omega_0 n}{2\alpha}}$$

$$k = \pm (1+i) \phi_n \text{ where } \phi_n = \sqrt{\frac{\omega_0 n}{2\alpha}} \quad (2.4)$$

$$k^2 = 2i\phi_n^2 \quad (2.5)$$

Now that we have solved for  $k$ , we can plug the value into equation 2.0 for the temperature distribution

$$T(x, t) = \sum_{n=0}^{\infty} e^{k^2 \alpha t} (A_n e^{kx} + B_n e^{-kx})$$

$$T(x, t) = \sum_{n=0}^{\infty} e^{2i\phi_n^2 \alpha t} (A_n e^{(1+i)\phi_n x} + B_n e^{-(1+i)\phi_n x})$$

$$T(x, t) = \sum_{n=0}^{\infty} \{A_n e^{\phi_n x} [e^{2i\phi_n^2 \alpha t + i\phi_n x}] + B_n e^{-\phi_n x} [e^{2i\phi_n^2 \alpha t - i\phi_n x}]\} \quad (2.6)$$

Utilizing the complex definitions again, to invert the exponentials in brackets

$$T(x, t) = \sum_{n=0}^{\infty} \{e^{\phi_n x} [A_n [e^{2i\phi_n^2 \alpha t + i\phi_n x}]] + e^{-\phi_n x} [B_n [e^{2i\phi_n^2 \alpha t - i\phi_n x}]]\}$$

$$T(x, t) = \sum_{n=0}^{\infty} \{e^{\phi_n x} [A_n \cos(2\phi_n^2 \alpha t + \phi_n x) + iA_n \sin(2\phi_n^2 \alpha t + \phi_n x)] + e^{-\phi_n x} [B_n \cos(2\phi_n^2 \alpha t - \phi_n x) + iB_n \sin(2\phi_n^2 \alpha t - \phi_n x)]\}$$

We can see that  $A_n$  will be eliminated since the temperature is finite when  $x$  is large. Then After utilizing the definition again of  $C_n$  with  $iB_n$  we find our temperature distribution to generally be

$$T(x, t) = \sum_{n=0}^{\infty} \{e^{-\phi_n x} [B_n \cos(2\phi_n^2 \alpha t - \phi_n x) + C_n \sin(2\phi_n^2 \alpha t - \phi_n x)]\} \quad (2.7)$$

Now we can apply our boundary condition of the temperature distribution at  $x$  equals zero,  $T(0, t) = T_0 \sin \omega_0 t$ , where  $B_0$  is eliminated due to only considering sine to find the coefficient  $T_0$ .

$$T(0, t) = T_0 \sin \omega_0 t = [B_0 \cos(2\phi_0^2 \alpha t - \phi_0(0)) + C_0 \sin(2\phi_0^2 \alpha t - \phi_0(0))]$$

We also apply that  $\omega_0 = 2\phi_0^2 \alpha$  and we find that  $T_0 = C_0$ . Applying the value of  $T_0$  to equation 2.7 and our definition of  $\phi_n$  from equation 2.4, we find the distribution of the junction temperature to be

$$T(x, t) = T_0 e^{-\phi_1 x} \sin(\omega_0 t - \phi_1 x)$$

$$T(x, t) = T_0 e^{-\sqrt{\frac{\omega_0}{2\alpha}} x} \sin\left(\omega_0 t - \sqrt{\frac{\omega_0}{2\alpha}} x\right) \quad (2.8)$$

Now we need to visually represent how the peak junction temperature, expressed as a fraction of  $T_0$  varies with the position of the junction. We can plot the distribution recognizing that to compare the max temperature ratio,  $\sin\left(\omega_0 t - \sqrt{\frac{\omega_0}{2\alpha}} x\right) = 1$  in equation 2.8. We can then move  $T_0$  to the other side of the equation to have our ratio.

$$\frac{T(x, t)}{T_0} = e^{-\sqrt{\frac{\omega_0}{2\alpha}} x} \sin\left(\omega_0 t - \sqrt{\frac{\omega_0}{2\alpha}} x\right), \text{ At Max Temp } \frac{T(x, t)}{T_0} = e^{-\sqrt{\frac{\omega_0}{2\alpha}} x} \quad (2.9)$$

To properly solve for the peak temperature expressed as a ratio of  $T_0$  as we move the junction from the motor, we must convert units of diffusivity from  $\frac{ft^2}{hr}$  to  $\frac{in^2}{hr}$  if we want to view the distance in inches. Here the thermal diffusivity of  $4.353 \frac{ft^2}{hr}$  converted for inches, would be  $626.832 \frac{in^2}{hr}$ . Plugging in all of our known variables, we can plot the junction location vs peak temperature expressed as a ratio of  $T_0$  as shown in Figure 3. We can see that due to the exponential form, the peak temperature ratio approaches zero as we move further and further from the motor.

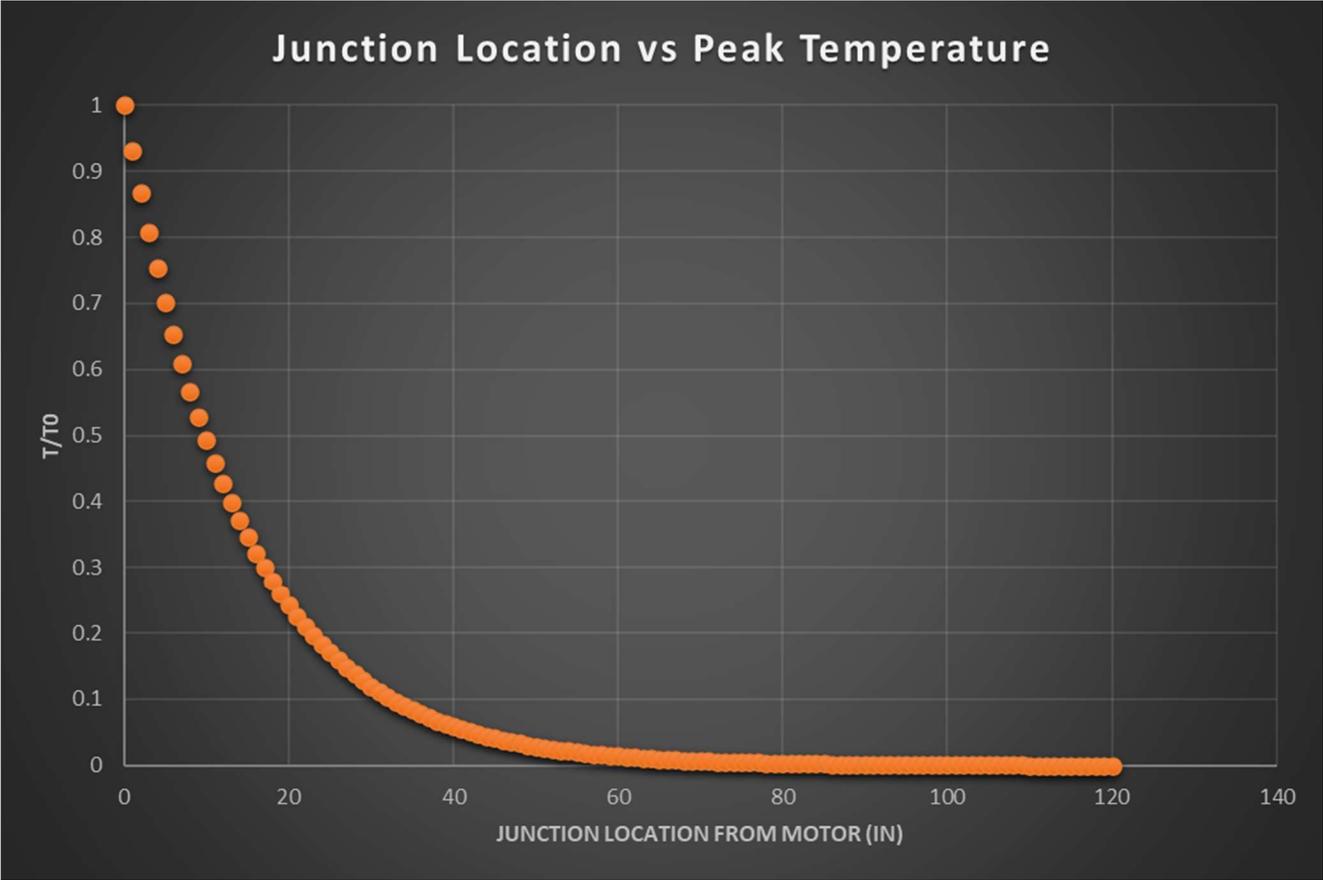


Figure 3: Junction Location vs Peak Junction Temp

Now instead of the temperature at the motor end being a sine wave, we observe when it is a square wave as shown in Figure 2. We would like to see how the temperature of the junction varies with time if the junction is located one foot from the motor. Observing the square wave in Figure 2, we can describe it as

$$T(0, t) = \begin{cases} T_1 & 0 \leq t \leq \frac{p}{2} \\ 0 & \frac{p}{2} \leq t \leq p \end{cases}$$

Using the structure of the Fourier series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

And knowing the coefficients using the Euler formulas are equal to

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, \dots$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, \dots$$

With a pulse wave as shown in Figure 2, we can show in Figure 4

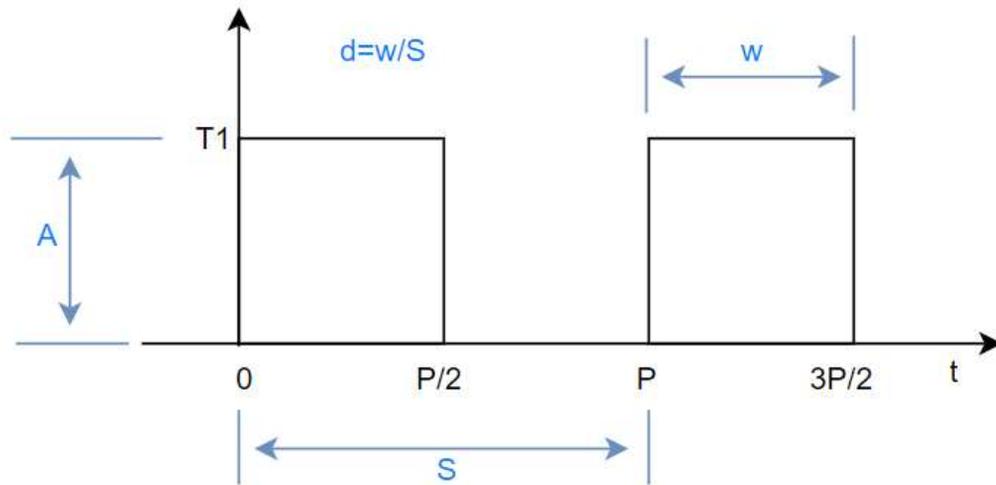


Figure 4: Pulse Wave Defined by Variables

With this wave, the coefficients for the Fourier series take the form

$$\begin{aligned} a_0 &= Ad \\ a_n &= \frac{2A}{n\pi} \sin(n\pi d) \\ b_n &= 0 \end{aligned}$$

In our case, plugging in our values given into the variables where  $S = P$ ,  $w = \left(\frac{P}{2}\right)$ ,  $d = \frac{P}{2}$ ,  $A = T_1$  then we obtain the Fourier series describing the wave as

$$T(0, t) = \left( \frac{P}{2} \right) T_1 + \sum_{n=1,3,5\dots}^{\infty} \frac{2T_1}{n\pi} \sin(n\omega_0 t)$$

$$T(\mathbf{0}, t) = \frac{T_1}{2} + \sum_{n=1,3,5\dots}^{\infty} \frac{2T_1}{n\pi} \sin(n\omega_0 t) \quad (3.0)$$

Now that we have found the Fourier series for the wave, again we can use the temperature distribution derived from the heat diffusion equation as described in equation 2.7 and set it equal to equation 3.0 at the instance of x equal to zero for our initial condition application.

$$T(0, t) = \frac{T_1}{2} + \sum_{n=1,3,5\dots}^{\infty} \frac{2T_1}{n\pi} \sin(n\omega_0 t) = \sum_{n=0}^{\infty} [B_n \cos(2\phi_n^2 \alpha t - \phi_n x) + C_n \sin(2\phi_n^2 \alpha t - \phi_n x)]$$

$$T(\mathbf{0}, t) = \frac{T_1}{2} + \sum_{n=1,3,5\dots}^{\infty} \frac{2T_1}{n\pi} \sin(n\omega_0 t) = B_0 \sum_{n=0}^{\infty} C_n \sin(2\phi_n^2 \alpha t) \quad (3.1)$$

To solve for the unknowns, we can look at equation 3.1 and match terms on both sides of the equation. This will show that the values of the coefficients and variables will be

$$B_0 = \frac{T_1}{2}, \quad C_n = \frac{2T_1}{n\pi}, \quad 2\phi_n^2 \alpha = n\omega_0, \quad \phi_n = \sqrt{\frac{n\pi}{\alpha p}}, \quad \omega_0 = \frac{2\pi}{p}$$

Plugging in our values defined above and expressing the temperature as a ratio of the distribution temperature over  $T_1$ , we have

$$\frac{T(x,t)}{T_1} = \frac{1}{2} + \sum_{n=0}^{\infty} e^{-\sqrt{\frac{n\pi}{\alpha p}} x} \left[ \frac{2}{n\pi} \sin\left(\frac{2\pi n}{p} t - \sqrt{\frac{n\pi}{\alpha p}} x\right) \right] \quad (3.2)$$

Our interest is in the temperature variation when the junction is located one foot from the motor. Our time units will be in minutes, so the thermal diffusivity is converted to  $\alpha = 0.07255 \text{ ft}^2/\text{min}$  and our period of 1 hour is converted to minutes which will equal 60 minutes. First, we will plot when the location is a zero feet away to see how well the Fourier Series approximates the initial condition of the square wave as we increase n. Then we will show the temperature wave at x is equal to one-half foot the temperature variation changes as we move away. Finally we show the temperature wave at x equals one foot. The Fourier series starts with n=1 and uses odd numbers but goes to infinity. To see how the temperature wave changes as we increase our n value we plot them all together.

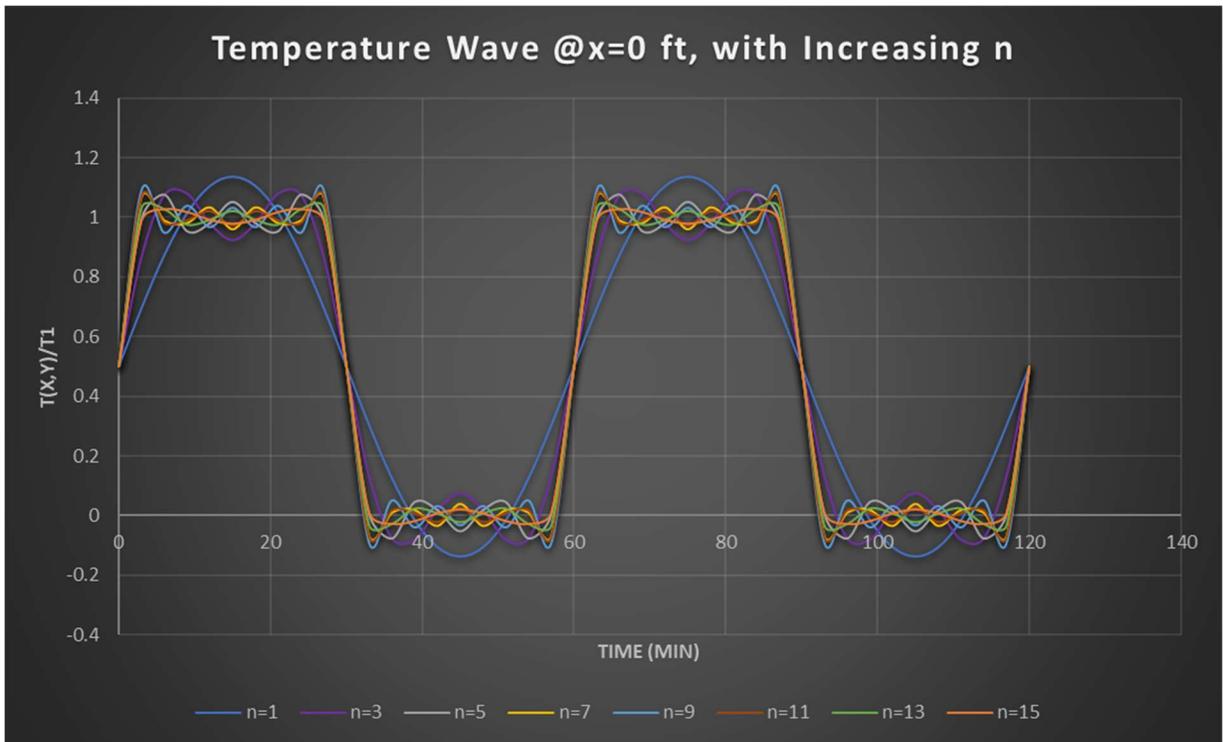


Figure 5: Temperature Wave at  $x=0$  ft w/ Increasing  $n$

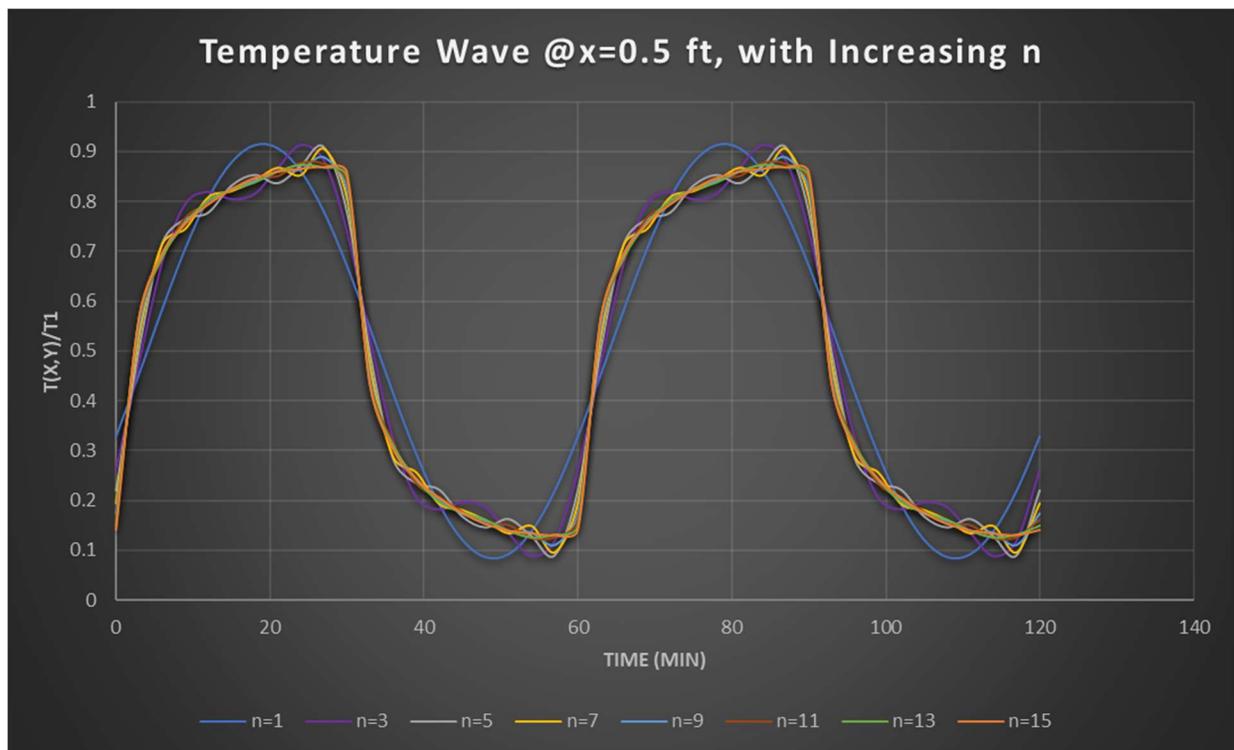


Figure 6: Temperature Wave at  $x=0.5$  ft w/ Increasing  $n$

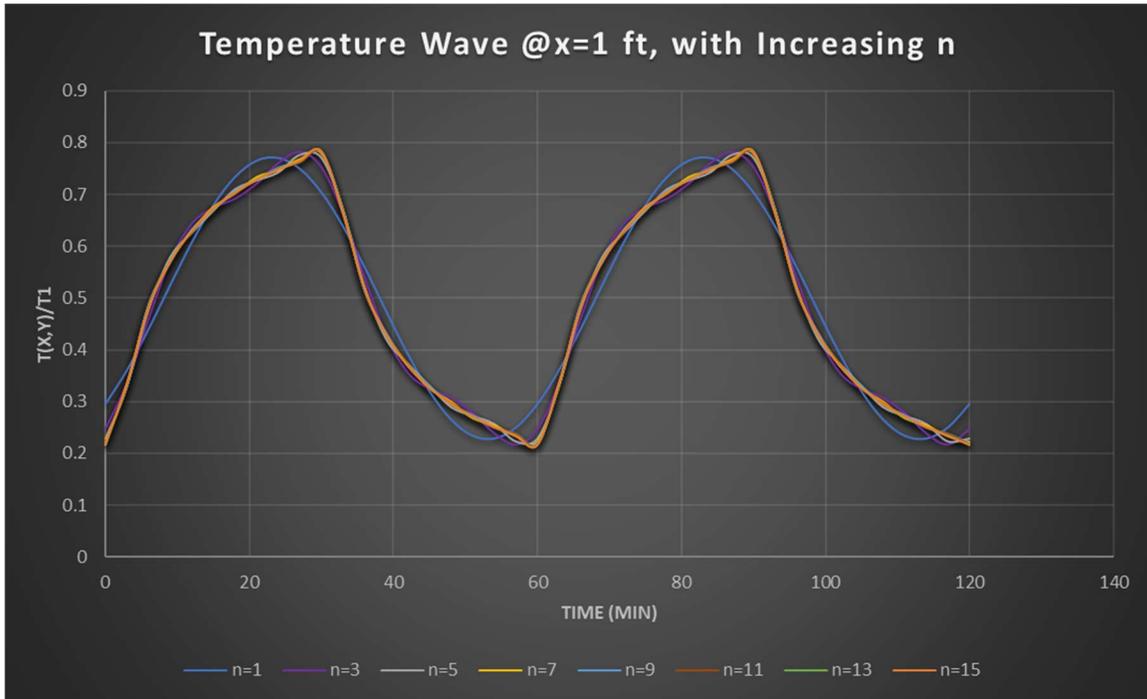


Figure 7: Temperature Wave at  $x=1$  ft w/ Increasing  $n$

As we increase  $n$ , we are approaching the square wave in the initial condition. Every increase in  $n$  is resulting in smaller changes, so we can view the temperature wave at  $n=15$  for all locations from zero feet to one foot.

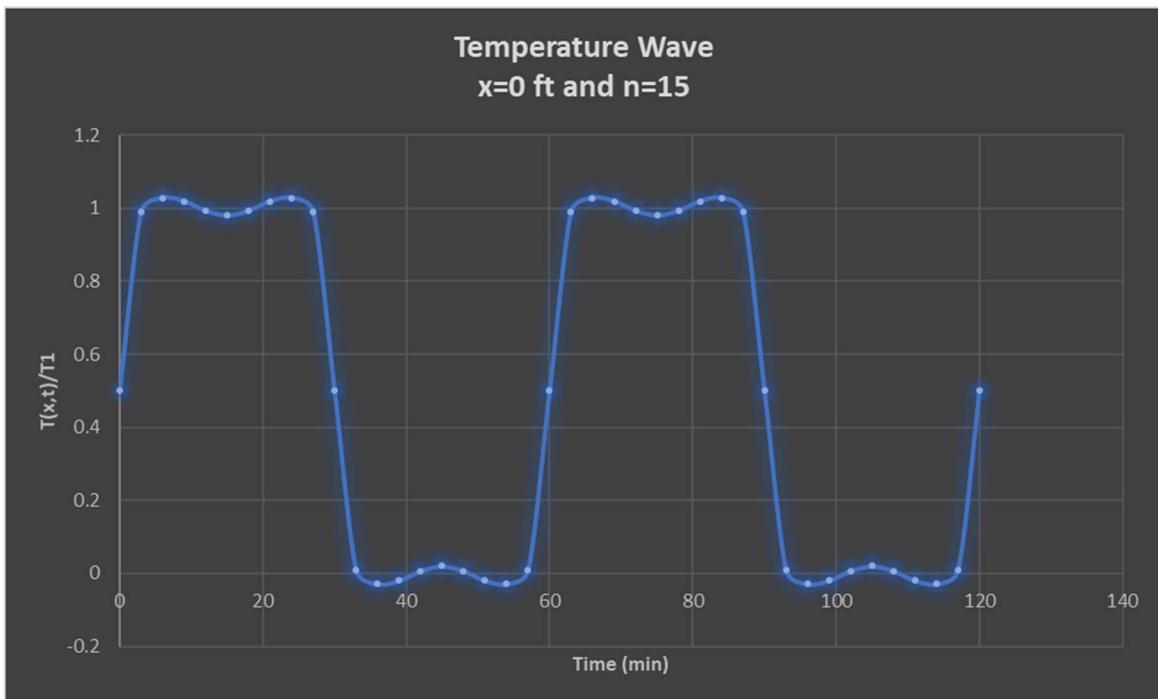


Figure 8: Temperature Wave at  $x=0$  ft and  $n=15$

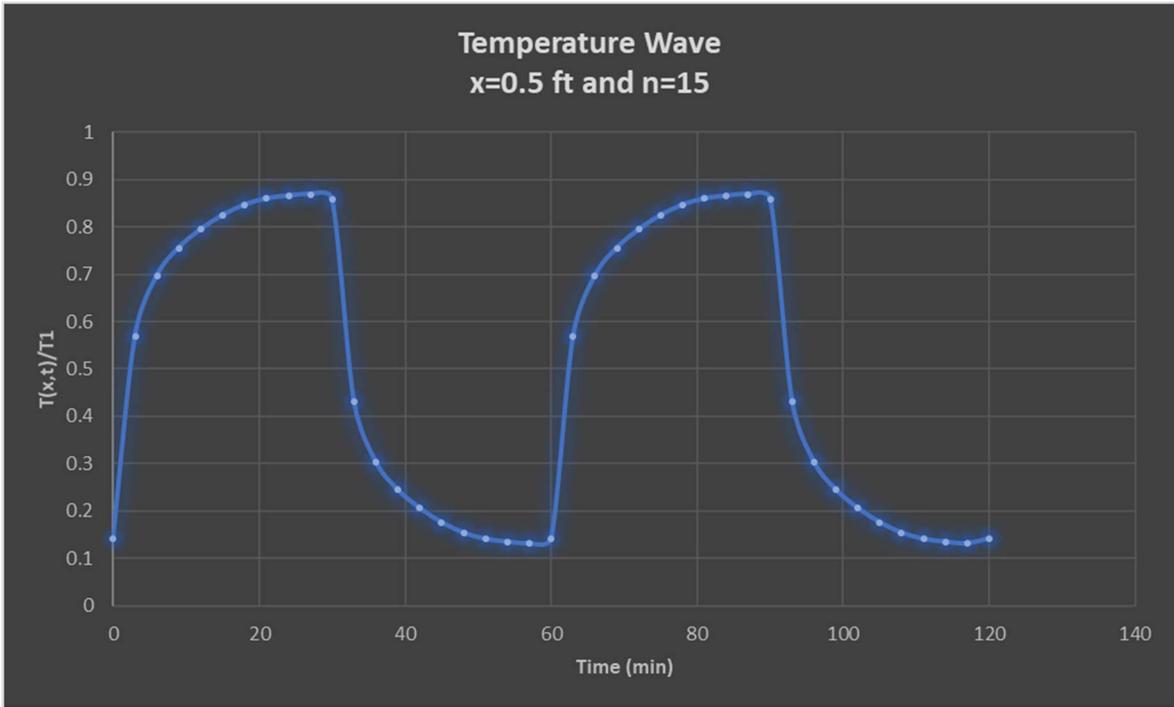


Figure 9: Temperature Wave at  $x=0.5$  ft and  $n=15$

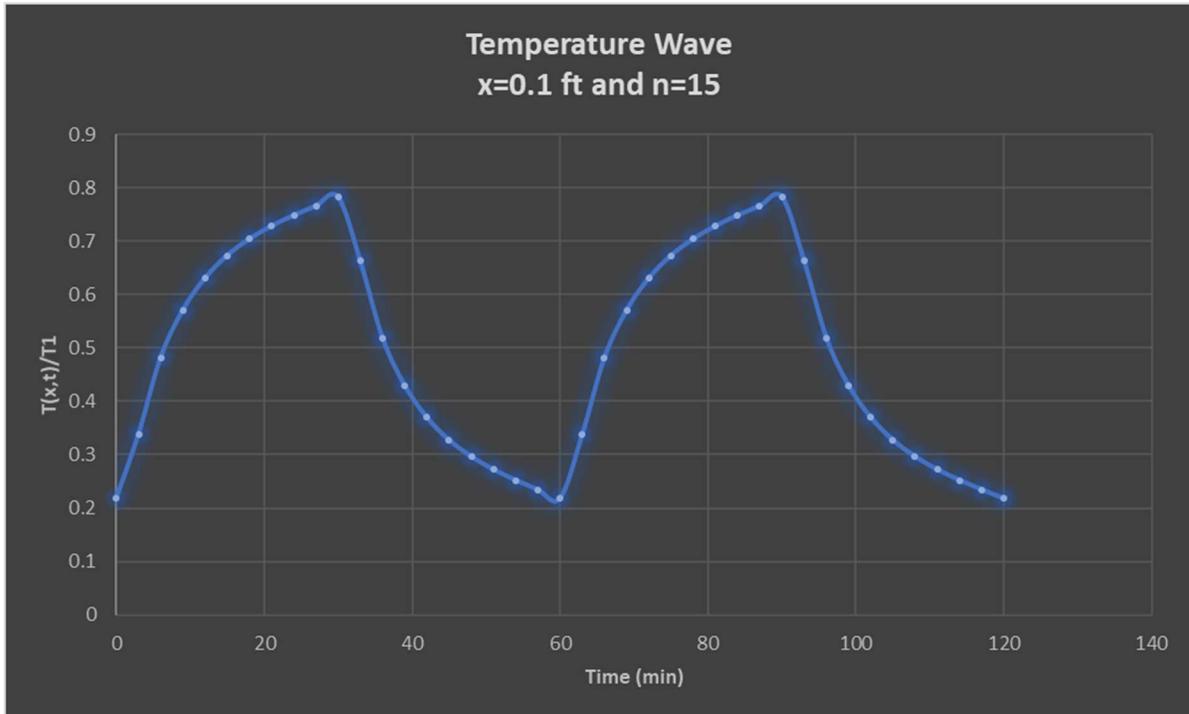


Figure 10: Temperature Wave at  $x=1$  ft and  $n=15$

The last portion of the problem that we need to analyze is if the waveform of the temperature wave is preserved as it moves along the cable and if not, why? To analyze this, the velocities of the first five components of the wave can be tabulated. In order to solve for the velocity, we need to solve for  $x$  at a particular event,  $k$  in our equation and take the first derivative. Then we can tabulate the first five components of the wave to show the velocity.

$$\frac{T(x, t)}{T_1} = \frac{1}{2} + \sum_{n=0}^{\infty} e^{-\sqrt{\frac{n\pi}{\alpha p}}x} \left[ \frac{2}{n\pi} \sin\left(\frac{2\pi n}{p}t - \sqrt{\frac{n\pi}{\alpha p}}x\right) \right]$$

$$k = \frac{2\pi n}{p}t - \sqrt{\frac{n\pi}{\alpha p}}x$$

$$x = \frac{\frac{2\pi n}{p}t - k}{\sqrt{\frac{n\pi}{\alpha p}}}$$

$$v = \frac{dx}{dt} = \frac{2\pi n}{p} \sqrt{\frac{n\pi}{\alpha p}}$$

$$v = 2 \sqrt{\frac{\alpha\pi}{p}} n \tag{3.3}$$

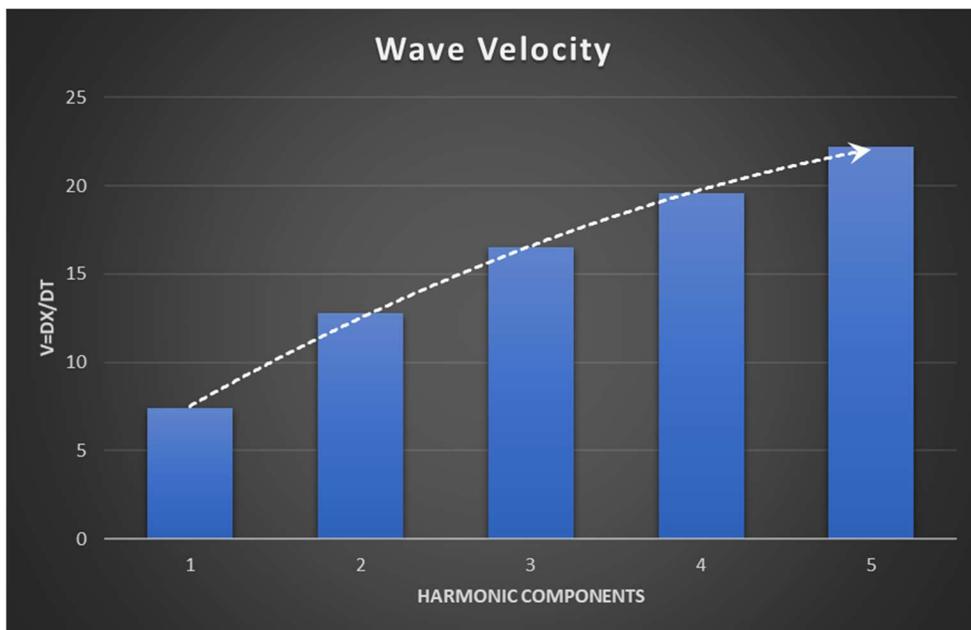


Figure 11: Wave Velocity,  $n=1, 3, 5, 7, 9$

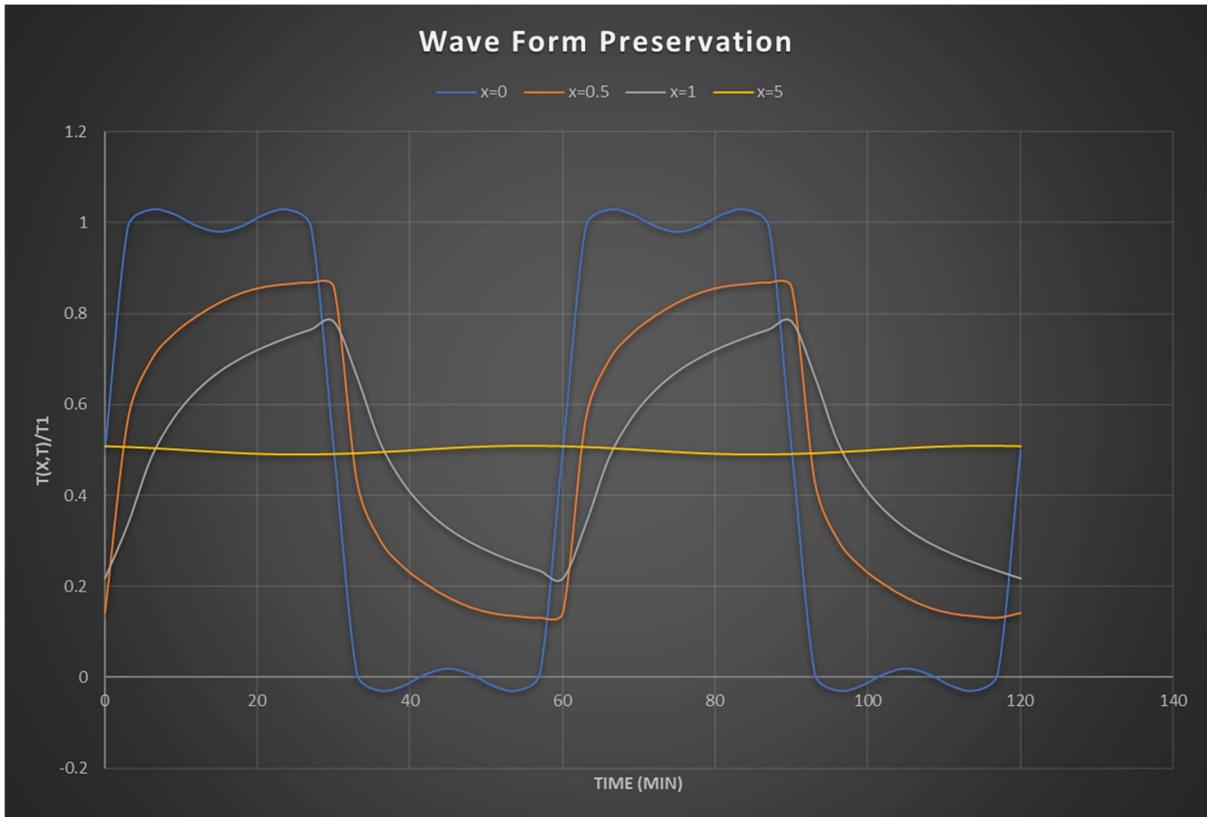


Figure 12: Wave Form Preservation Along Cable

Though the velocity increase can help explain the wave form preservation, we can also plot the wave forms over each other as they move further from the motor. Figure 12 shows the wave form when  $x=0$  ft, 0.5 ft, 1 ft, and 5 ft. We can see that the wave form is not preserved as we move away from the motor.

## Discussion

The mathematical methods used in this report and analysis prove to be complex when dealing with a partial differential equation but can be converted to more simple forms that can be analyzed using earlier methods of ordinary differential equations. This particular problem lends itself well to the heat diffusion equation since we are interested in the temperature distribution in a transient, one-dimensional problem. The final distribution is shown in equation 2.7 and can be utilized for differing initial conditions.

The steps involved in analyzing the problem includes identifying the appropriate partial differential equation, use of separation of variables method, find an appropriate separation constant, apply the homogenous boundary conditions, utilize the superposition principle, and then use trigonometric orthogonality conditions to apply the non-homogeneous boundary conditions. This will allow us to solve for the coefficients in the temperature distribution and solve for the final distribution when faced with differing conditions such as the sine or square wave at the motor. We can also see in the derivation of the use of converting to and from the complex form which allows us to manipulate the equations in a manner much easier than with periodic functions. Using exponentials allows the flexibility to use identities to convert back and forth and not worry as much about sine and cosine rules.

Of relevance to the dissipation of the temperature distribution as the junction is moved further and further down the cable, is the wave velocity. The wave is technically a distribution that moves from one end of the cable to the other. The distance in which the crest of the wave travels can be observed, while the change in position over time is observed. We see that depending on the medium that the wave moves through, the wave speed can be affected. In this case, we have a constant medium of constant diffusivity through the cable. It is not affected by the wavelength or the frequency. The further the wave travels through a medium, the more it will decay the properties of the wave.

From this study, we can also discuss first and second order system responses. First order differential equations can be used to describe many systems in engineering. An example of this could be in the transient response of a thermocouple. When the thermocouple is subject to a disturbance in the form of a step input, we can see that the rate that the response of the thermocouple approaching the final value is decided by the time constant. If there is a transient system where the thermocouple is subjected to a sudden temperature change, then it will take some time to respond to the change. This becomes important because if the response time of the thermocouple is too slow, then it may not capture the rate of change of temperature that a system is trying to measure.

With a second order system, we can see energy storage and energy exchange like in a mass spring system or some kind of transducer such as a pressure transducer. The responses of these systems depend on the dissipative elements of the system. We have studied this extensively in the mid-semester project with the vibration isolation for a drop forge. Here we studied a system with differing damping ratios ( $0, >1, 1, < 1$ ). These cases are referred to as undamped, overdamped, critically damped, and underdamped, respectively. The damping ratio in this case is very critical to have an efficient system that is not over or under sensitive. A system with no damping, would theoretically oscillate forever. Since all systems have at least a little bit of damping by nature, we can view an undamped system by taking a picture of the system motion for a small amount of time. For a mass spring system, we can view the harmonic oscillations that are essentially the outside forces that restore the system to its equilibrium point after being displaced. The harmonic oscillation has a constant amplitude and a sinusoidal motion associated. With a damped system we can see different changes that may occur. When the system is overdamped, we have distinct real roots for the characteristic equation. From the case name we can see that this type of damping removes energy in the system at a high rate from the system which does not allow oscillations and steady state is achieved slowly. This case is also known as a non-oscillatory state. An underdamped system yields complex conjugate roots and the amplitude of the oscillations start out large and steadily decline until they reach zero. In the critically damped system, the system quickly returns to equilibrium or steady state without any oscillations, which can be viewed as a damping case in between the under and overdamped cases. In this case, the initial velocity can cause overshoot if a non-zero value.<sup>4</sup>

Ultimately, we were able to present the most information to the manufacturer to reduce the chances of the metallurgical phenomenon occurring at the junction of the cable. With this information from the analysis of a partial differential equation, a decision can be made based upon the design envelope of the system. We could recommend a location for the junction depending on the driving function at the motor, but we may be out of the bounds of the design window. The information presented in the graphs, analysis, and conclusion gives ample data to support decisions made moving forward while the material engineers work out the reasoning for the metallurgical phenomenon.

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